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TIME-OPTIMAL RESPONSE OF FEEDBACK CONTROL SYSTEMS BY MEANS OF DISCONTINUOUS RATE COMPENSATION

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Determining the time-optimal response of a feedback control system is a problem vital to many applications of such systems. This problem consists of obtaining the fastest possible output response to some stimulus without exceeding the physical constraints placed on the system and its components. Of the schemes used to optimize the time response of control systems, the most widely used and studied has been the "bang-bang" solution. The bang-bang approach, however, has one major physical disadvantage; namely, that the drive on the controlled plant must be turned off at the exact instant the final state of the system output is reached, and that it then remains turned off. To overcome this disadvantage, the use of discontinuous rate compensation, where an "ideal" relay is placed in the tachometric feedback loop, is here investigated as a means of obtaining time-optimal response of a feedback control system.

The primary purpose of this investigation is to study the time response of a discontinuous rate compensated, or d. r. c., system and to compare this response with that

achieved using a bang-bang configuration of the same system. This purpose was accomplished by analyzing the system configurations mathematically, devising analog models of those configurations, programming the models on an analog computer and obtaining output response data from the analog computer simulations.

It was found that the best method among those attempted for studying time-optimization of feedback control systems is analog computer simulation of the systems. Both digital computer and analytical methods were found to require a considerable amount of algebraic tedium. It was concluded from the data obtained in this investigation that the d. r. c. configuration of the basic system chosen, switched in such a way that its response overshoot was always a specified small amount, exhibited better overall time response than any other configuration studied in terms of rise time and percent overshoot. Suggestions were made for future studies.

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CHAPTER I

INTRODUCTION

I. STATEMENT OF THE PROBLEM

Selecting the best, or optimum, response for a given feedback control system with respect to some criterion is a problem vital to many applications of such systems. Often, the criterion selected for optimization is that of response time. In such a case, an attempt is made to obtain the fastest response of a system to some stimulus without exceeding physical constraints placed upon system components.

Several schemes for optimizing the time response of feedback control systems have been attempted earlier by other investigators. Probably the most widely used and studied of these methods has been the so-called "bang-bang" approach[1]. In a bang-bang system, an ideal relay is placed in the forward loop of the system in such a way that it can reverse the direction of the drive on the plant. The relay is operated such that minimum response time is obtained. The bang-bang method of achieving optimum time response is developed in Chapter II.

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[1]John E. Gibson, Nonlinear Automatic Control (New York: McGraw-Hill Book Company, 1963), p. 126.

There is one great drawback in using bang-bang control; i.e., at the instant the final state of the output is reached, the drive on the plant must be turned off because otherwise, as will be shown, the system steady-state error never approaches zero. Also, if the switching point does not occur at exactly the right instant, the final output state will not be reached. For this reason, another method of achieving optimum time response was sought. The method studied in this investigation is that of discontinuous rate compensation. In a discontinuous rate compensated, or d. r. c., system, an ideal relay is placed in the tachometric feedback loop. In a d. r. c. system, when the plant output reaches its final value, the system drive power supply need not be turned off to maintain this final state, as is the case with bang-bang control. Also, since the final value is always reached, the switching point in a d. r. c. system is not critical, as it is in a bang-bang system.

II. SCOPE OF THE INVESTIGATION

The primary purpose of this investigation is to study the time response of a d. r. c. system and to compare this response with that achieved using a bang-bang configuration of the same system. This purpose is accomplished in the following five steps: (1) presenting mathematical system analysis, (2) building analog models, (3) performing simulation studies and obtaining data, (4) analyzing the data, and (5) drawing conclusions and offering suggestions for extension of the study.

A mathematical system analysis is presented. In Chapter II, the response of the system under consideration in this study is analyzed in its basic configurations using conventional control system theory and time-optimal control theory. The time response of the basic system to a unit step input is determined using normalized response curves and by solving the system output time equations on a digital computer. The digital computer solutions are then plotted and are later used to check the validity of analog computer models of the system. Time-optimization of control system response using switching of the control effort is then presented. The bang-bang solution is included along with a discussion of optimum switched systems in terms of the phase plane.

Analog computer models are constructed. In Chapter III, a general presentation of analog computer programming principles is made. Computer models of the basic system, the basic system with tachometric feedback, the bang-bang system and the d. r. c. system are then constructed. The time responses of the basic system models, with and without tachometric feedback, are compared with those obtained by means of a digital computer in Chapter II. The validity of the bang-bang and d. r. c. models is thus established.

Simulation studies are performed and data is obtained. In Chapter IV, a presentation is made of the analog computer studies performed on the various system configurations. System response times to steps of various magnitudes are obtained from strip-chart recordings of system outputs. Other response criteria, such as percent overshoot, maximum output velocity and maximum output acceleration, are obtained for phase portraits of each system configuration. The data presented in Chapter IV includes tabulations of response times, overshoot percentages, maximum output velocities and maximum output accelerations for each system configuration as a function of input step magnitude. Also included are phase portraits of each configuration as produced by an x-y plotter.

In Chapter IV, the information content of the data presented therein is evaluated and discussed. Response characteristics and the use of switching curves are among the topics presented. The relative merits of the various system configurations studied are discussed.

In Chapter V, conclusions are drawn regarding this study in general and the response of the d. r. c. system in particular. Possible improvements in the method of study are cited. Suggestions are made of possible extensions of this study based on the results and experience obtained. Areas such as switching function generation and digital computer model building are suggested.

CHAPTER II

MATHEMATICAL SYSTEM ANALYSIS

Using conventional control system theory, the response of a basic system to a step input was determined. The first attempt at improving the response of the system was made by adding tachometric feedback to the plant. It was found that, in the first case, the time response was fast but had excessive overshoot. In the second case, the overshoot problem was solved at the expense of the response time. The conclusion reached was that some other method of improving the system response should be attempted.

A mathematical investigation of the time-optimal response of the system was then made. This investigation led to the conclusion that optimum time-response, based on certain assumptions, could be achieved by using "bang-bang" control; that is, by using an ideal reversing switch in the forward loop. Chapters III, IV and V of this thesis involve a study of the time-response of a system using discontinuous rate compensation and a comparison of this response with that of the same system in a bang-bang configuration.

I. CONVENTIONAL ANALYSIS OF SYSTEM TIME-RESPONSE

The basic system and the basic system with tachometric feedback were both analyzed mathematically for response time and percent overshoot. Two sets of system constants were used in each case to broaden the scope of the investigation. In both cases, the system time solution was obtained with the help of a digital computer to perform the numerous calculations involved. It was found that, though the addition of tachometric feedback reduced the overshoot considerably, the response time was lengthened by a significant amount. The conclusion reached was that some method other than simple tachometric feedback must be used to improve system response.

Basic System

A block diagram of the basic system is shown in Figure 2.1. The Laplace transform of the plant transfer function is $G(s) = K_1/s(s+b)$. The system input, $r(t)$, is a step function whose magnitude will be taken to be unity for the sake of simplicity. The transformed input is then $R(s) = 1/s$. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} \quad \frac{K_3G(s)}{1 + K_3G(s)}.[1] \tag{2.1}$$

[1] Robert N. Clark, Introduction to Automatic Control Systems (New York: John Wiley & Sons, Inc., 1962), p. 180.

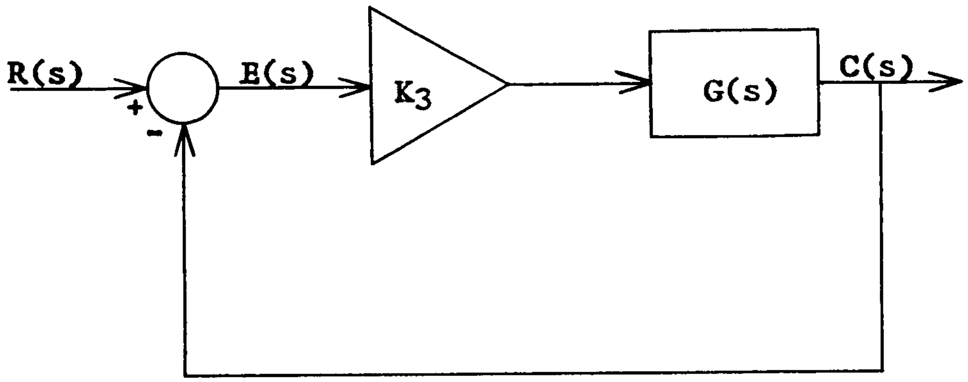

Figure 2.1. Basic System.

Substituting the expression for G(s) into equation (2.1) yields the following closed-loop system transfer function:

$$\frac{C(s)}{R(s)} = \frac{K_1K_3}{s^2 + bs + K_1K_3} \tag{2.2}$$

The natural resonant frequency, w_n, and the damping ratio, ζ, of the system are found from the standard second-order form[2],

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}\,. \tag{2.3}$$

In order to broaden the scope of this investigation, as well as to study the effects of varying system parameters on output performance, two sets of constants were used. The effects on the system time response of using either set of values were studied for all system configurations. To avoid confusion, the nomenclature "system #1" and "system #2" was used to distinguish between the two sets of values. In system #1, the following set of parameters was used: $K_3 = 1.42$ and $b = 2$. In system #2, the following values were used: $K_3 = 1.62$ and $b = 3$. In both systems, $K_1 = 10$. These values were chosen for mathematical simplicity as well as to provide a basic system whose output response was clearly underdamped and required some improvement. The primary difference between system #1 and

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[2]Clark, op. cit., p. 111.

system #2 is in the magnitude of the drive to the plant, although the plant transfer functions were made somewhat different also.

If numerical values are substituted into equation (2.2), then values of natural resonant frequency and damping ratio for the two systems are found from the standard-form equation (2.3). For system #1, $w_n = 3.77$ radians per second and $\zeta = 0.265$. For system #2, $w_n = 4.03$ rad./sec. and $\zeta = 0.372$. These values can be used to find percent overshoot and rise time of the system output from normalized, second-order curves[3]. The expression "rise time", as it is used in this thesis, means time from initial output condition to the first time the output comes within 5% of the final condition. It should be noted that this is not the standard definition of rise time. For system #1, % overshoot = 42% and rise time = 0.45 sec. For system #2, % overshoot = 28% and rise time = 0.5 sec.

The experimental data obtained in this study consists of analog computer outputs. Since these outputs are continuous time functions of the system outputs, it would be desirable to express the results obtained in this section as continuous time functions also. These time functions can then be used to validate the results

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[3]Clark, op. cit., p. 69.

obtained from the analog models constructed in Chapter III. Computing somewhat complicated time functions, such as were found in this study, so that they may be plotted is tedious and potentially inaccurate. For this reason, the time functions were calculated using a digital computer.

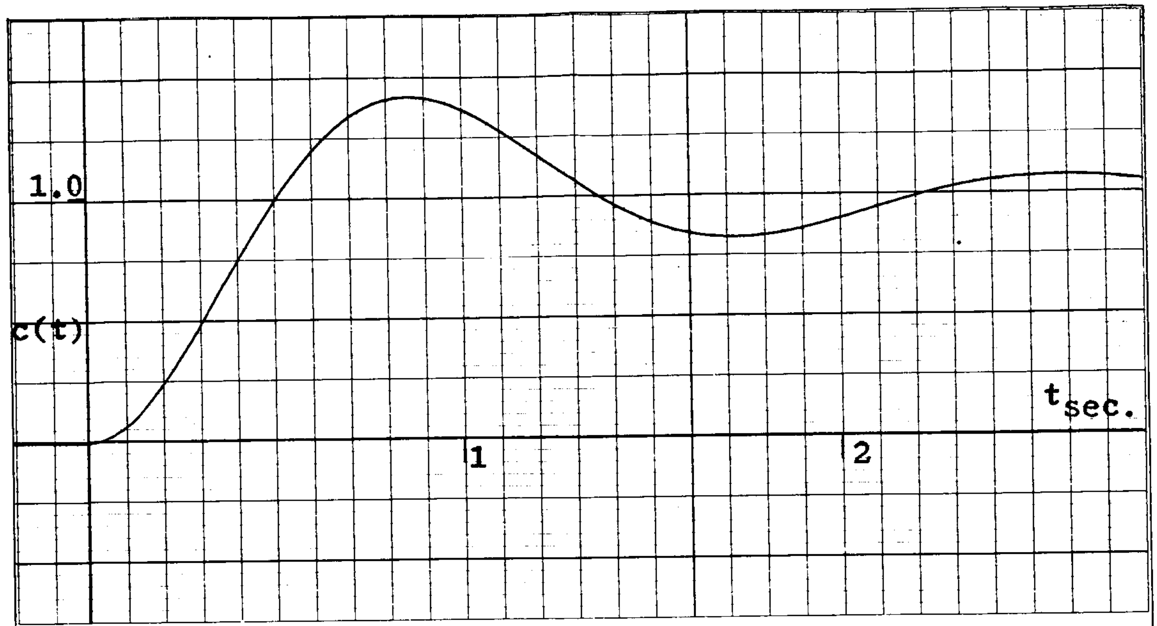
The actual output time function, c(t), was found from the time-response equation. The time-response equations for systems #1 and #2 were found by substituting numerical parameter values into equation (2.2) and solving the equations for C(s), where R(s) = 1/s. These equations were then broken up into partial fractions, and the inverse Laplace transform of each fraction was taken. The output time function of system #1 was found to be

$$c(t) = u(t) - 1.04\exp(-t)\sin(3.63t+1.302) \tag{2.4}$$

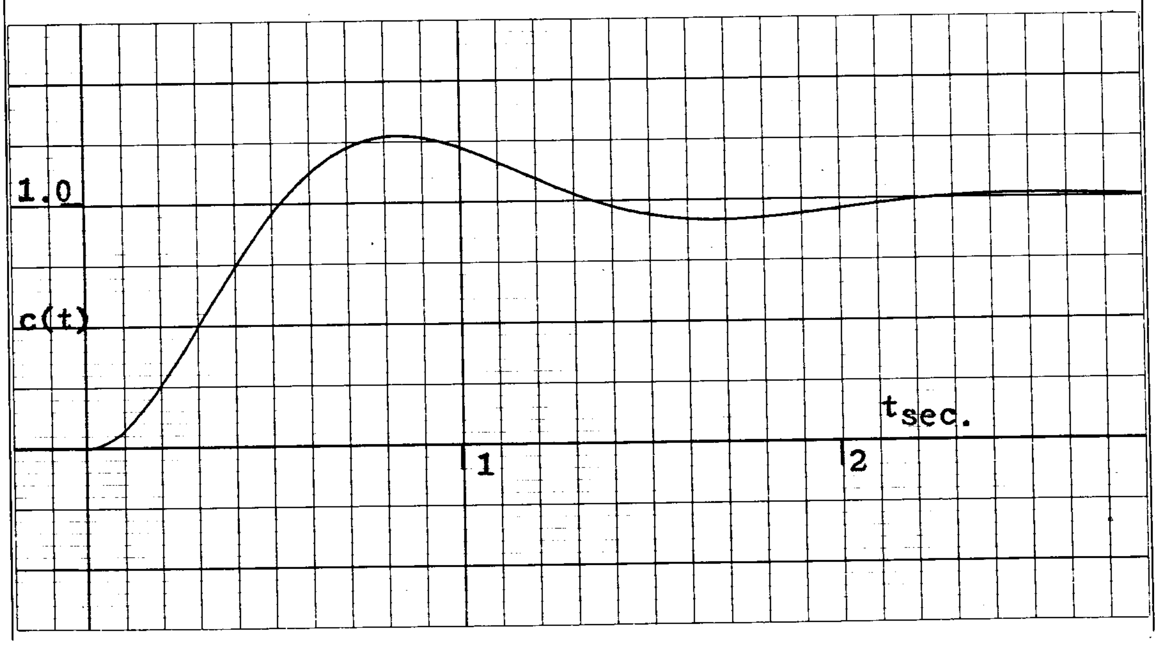
Similarly, the output of system #2 was found to be

$$c(t) = u(t) - 1.08\exp(-1.5t)\sin(3.735t+1.188) \tag{2.5}$$

Equations (2.4) and (2.5) were evaluated on a digital computer, the program for which is included in Appendix A. Calculations were made for the period $0 \leq t \leq 3$ seconds at intervals of 0.05 second. The output functions thus obtained are shown in Figure 2.2. From the curves in Figure 2.2, the percent overshoot and rise time of system #1 were found to be 42% and 0.48 sec. For system #2, these values were found to be 29% and 0.5 sec. These



(a) system #1



(b) system #2

Figure 2.2. Analytical Output response of basic system.

values all agree well with those found from the second-order response curves, providing a good check for the digital computer program and the curves obtained thereby. The curves in Figure (2.2) were used to validate the analog computer models developed in Chapter III and to check their output accuracies.

For many practical applications, an overshoot of 42% or of 29% in system response is excessive and unacceptable. It is apparent that a means of increasing system stability, i.e., decreasing the tendency of the output to oscillate, is desirable in such a case. A common means of stabilizing a control system is to introduce a tachometric feedback loop. The general effect of adding tachometric feedback to a system is one of limiting the output velocity. Moreover, the mathematical analysis of a system is not made more difficult with the addition of tachometric feedback. Therefore, this method of improving the system response was the first attempted. The failure of this method to provide an acceptable system output response in either system #1 or in system #2 prompted, in part, the study of constraints in the system. This effect of constraints is the subject of study in the remainder of this thesis.

Basic System with Tachometric Feedback

The block diagram of the basic system with the addition of tachometric feedback is shown in Figure 2.3. The values of K_3 and G(s) for both systems are unchanged from the previous section. The Laplace transform of the tachometric feedback is $H(s) = K_2 s$, where $K_2 = 0.4$. This value of K_2 was chosen to yield a particular value of damping ratio for system #1, and was also used for system #2.

The system with tachometric feedback was studied in exactly the same manner as was the basic system of the previous section. For this reason, the detailed analysis of this system configuration will not be presented. The closed-loop transfer function of the system with tachometric feedback was found to be

$$\frac{C(s)}{R(s)} = \frac{K_1K_3}{s^2+(b+K_1K_2)s+K_1K_3} \tag{2.6}$$

Equation (2.6) is also in the second-order form of equation (2.3). For this configuration, system #1 was found to have a natural resonant frequency w_n of 3.77 rad./sec. and a damping ratio of 0.80. For system #2, $w_n = 4.03$ rad./sec. and $\zeta = 0.87$.

As would be expected, the addition of tachometric feedback had no effect on the natural resonant frequency of either system; the entire effect was felt by the damping ratio, which was increased substantially in both cases. From the second-order response curves cited

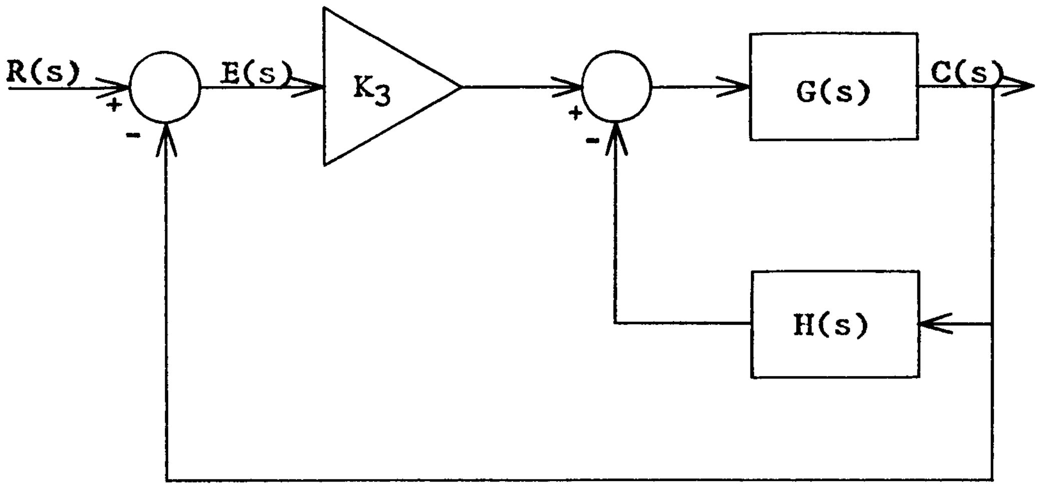


Figure 2.3. Basic System with Tachometric Feedback.

earlier, the percent overshoot and rise time for system #1 were found to be 2.5% and 0.86 sec. For system #2, these values were 1% and 1.25 sec.

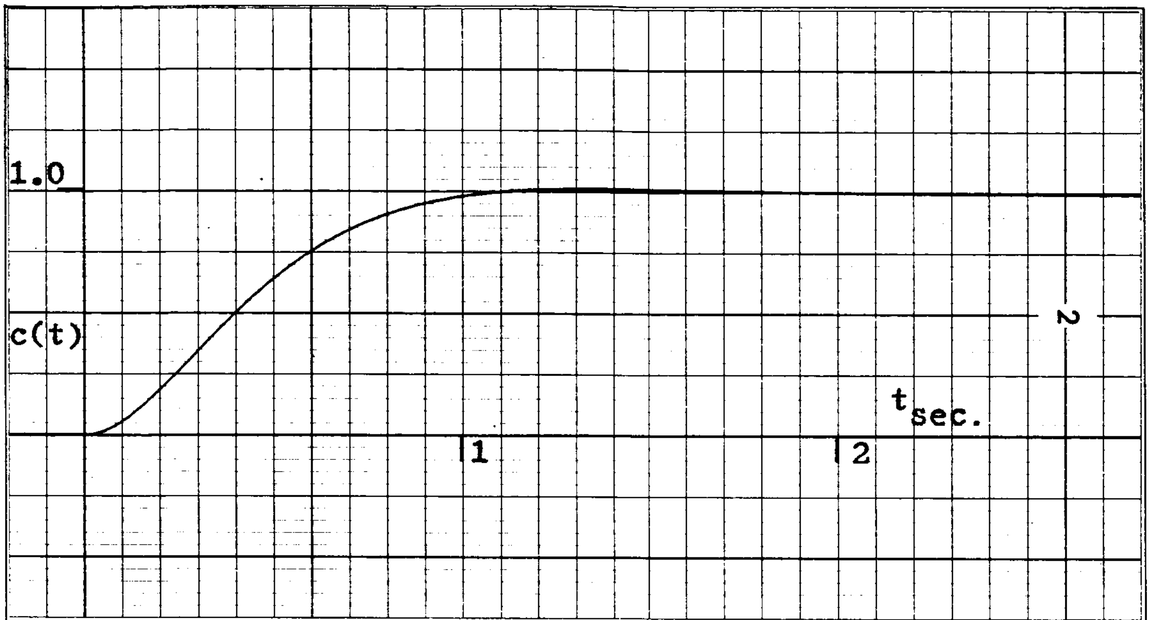
Because of the increase in damping, the effect of adding tachometric feedback was to decrease the overshoot and to lengthen the response time. Although the overshoot is now at a more physically acceptable level, the response times have been approximately doubled. Using the same procedure as in the previous section, the output time functions of this system configuration were found to be, for system #1,

$$c(t) = u(t) - 1.739\exp(-3t)\sin(2.236t-2.501). \quad (2.7)$$

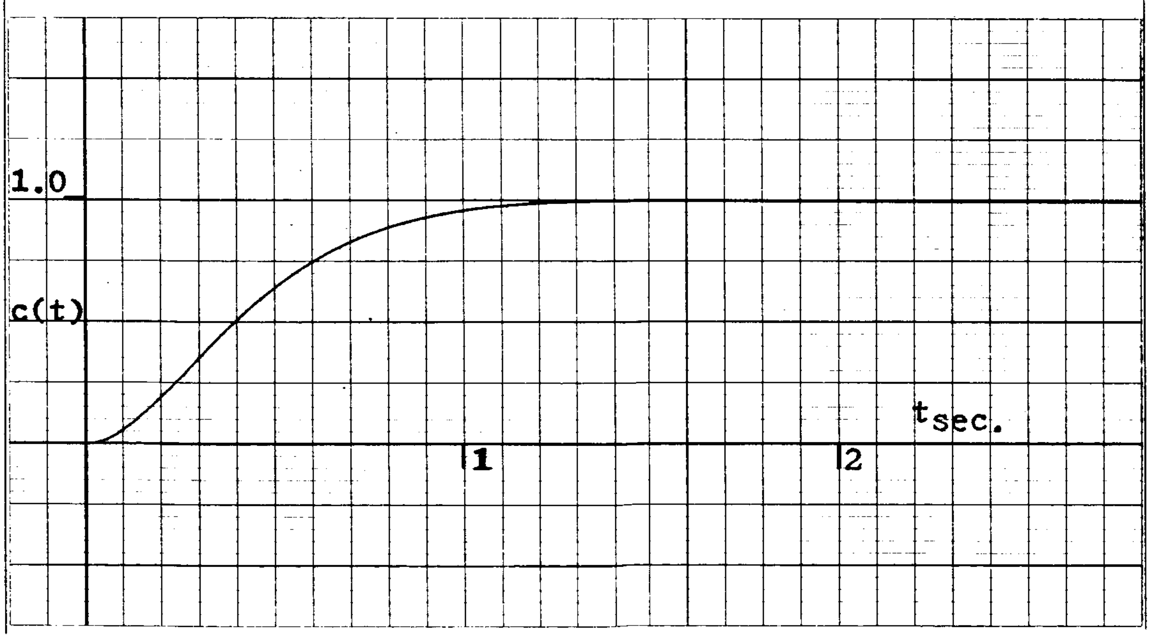
For system #2,

$$c(t) = u(t) - 2.02\exp(-3.5t)\sin(1.987t+0.517). \quad (2.8)$$

These functions were also calculated using a digital computer for $0 \leq t \leq 3$ sec. at intervals of 0.05 sec. The output functions were plotted and the resulting curves are shown in Figure 2.4. From these curves, percent overshoot and rise time were found to be 2.5% and 0.88 sec. for system #1, 1% and 1.25 sec. for system #2. Again, these values agree well with those obtained from the normalized curves, providing a good check on the digital computer program. The curves in Figure 2.4 were also used to validate the analog models of Chapter III and to check their accuracies.



(a) system #1



(b) system #2

Figure 2.4. Analytical output response of basic system with tachometric feedback.

Conclusions

Two sets of conclusions can be drawn from the results of the study of this basic control system in its various configurations. The first conclusions concern the behavior of the system itself with respect to time response and overshoot. The second set of conclusions concerns the method of investigation, which has been conventional, straightforward and somewhat tedious for the amount of information derived.

Conclusions were drawn regarding system behavior. In the case of the basic systems, the overshoot and subsequent oscillation were found to be excessive, though the response times might be considered sufficiently short. When tachometric feedback was introduced, the damping ratios were increased, reducing the amounts of overshoot to desirable levels, but approximately doubling the response times in the process. A method of optimizing response time while keeping overshoot small would then be desirable. One method, utilizing a discontinuity in the system control was studied. Part II of the present chapter is an investigation of configurations utilizing such discontinuities.

Conclusions were also drawn regarding the method of investigation. Although the use of a digital computer eliminated the tedium involved in manually solving the time equations repeatedly to obtain system time responses, finding the equations in the first place involved no small amount of algebraic manipulation. Also, finding the maximum values of velocity and acceleration to which a given system configuration is subjected would require considerable additional effort. If these difficulties are coupled with the fact that manual computation of the time response of a system with even the simplest non-linearities becomes next to impossible, a strong case is made for the use of some other method of investigation. The method of analog computer model-building was used extensively in this study to simulate systems with dis-continuities, and is described in Chapter III.

II. TIME-OPTIMIZING CONTROL SYSTEM RESPONSE

The problem of time-optimizing the response of feedback control systems has been studied extensively. A time-optimal solution, called the "bang-bang" solution, was developed and has been known since the early 1950's. Time-optimal control theory has been applied to this solution, and was presented by Athanassiades[4]. The bang-bang solution involves the use of an ideal switch in the forward loop of the system. In this section, a brief description of the bang-bang solution is presented. Also included is a description of optimum switched systems utilizing phase plane techniques. Finally, reasons for choosing a discontinuous rate compensated system configuration for study are presented.

Bang-Bang Time-Optimal Solution

In the development of the bang-bang solution through the use of optimal control theory as presented by Athanassiades[5], an optimal control law for minimum time response is derived. Mathematically, this law can be stated as

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[4] Michael Athanassiades, "Optimal Control for Linear Time Invariant Plants with Time, Fuel and Energy Constraints", Applications and Industry, Number 64 (New York: The Institute of Electrical and Electronic Engineers, January, 1963), p. 322.

[5] Athanassiades, Ibid., p. 322.

$$u_j(t) = -\mathrm{sgn}\left[q_j(t)\right], \tag{2.9}$$

where $u_j(t)$ is the "jth" term of an nth-order control function, $q_j(t)$ is the jth term of an rth-order function of the system costate vector, and the term "sgn", or "signum", indicates

$$-\mathrm{sgn}\left[q_j(t)\right] = -\frac{q_j(t)}{|q_j(t)|}. \tag{2.10}$$

The following conclusions can be drawn from the optimum control law:

1. The optimum control signal for minimum time operation is piecewise constant.
2. The optimum control signal must have only the values of +1 or -1.
3. The polarity of the control signal depends on the output of the adjoint system.[6]

The exact time equation of the optimum control function $u^o(t)$ for an nth-order system is not presently known and can be derived for a second-order system only with a considerable amount of algebraic tedium. Moreover, the development of $u^o(t)$ assumes sets of initial conditions, and is thus limited. However, this tedium can be avoided through a systematic trial-and-error procedure using analog computer simulation of a bang-bang system This simulation was carried out in the present study using the analog model developed in Chapter III. The results of this simulation are presented in Chapter IV.

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[6]Athanassiades, op. cit., p. 322.

Optimum Switched Systems

Before considering response time optimization through the use of switched systems, it would be useful to describe concepts of the phase plane briefly, since the phase plane is a very useful tool in describing such systems. The phase plane is a two-dimensional space on which the first derivative of a variable is plotted versus the variable itself. For a closed-loop system, either the error or the system output and its rate of change are plotted. Phase plane analysis is limited to second-order systems, since derivatives of higher-order than the first cannot be represented on the phase plane, and higher-order systems would not be completely defined by a variable and its first derivative.[7] Use of the phase plane in this thesis will be confined to plots of the system output velocity versus output displacement.

Facility in visualizing physical occurrences as they are described in the phase plot, or trajectory, is obtained only after considerable exposure to and study of various system phase "portraits", or complete trajectories. Initial conditions of second-order system variables are represented as points which can theoretically occur anywhere in the phase plane. The direction in which a phase trajectory travels depends on the system stability; the

[7] John E. Gibson, Nonlinear Automatic Control (New York: McGraw-Hill Book Company, 1963), p. 237.

shape of the trajectory depends on the system parameters and configuration. Time, though it does not appear as a variable in the phase plane, is implicit in a trajectory and can be recovered. In general, time is inversely proportional to the area under the trajectory, with reference to the axis of the variable; in this case, the variable under consideration is the output displacement. The phase plane concept will be used extensively throughout the remainder of this thesis.

When considering time-optimization of system response by means of switching one of the system drives, the phase plane representation of system response becomes a useful tool for study as well as for presentation. Assume a simple, second-order positional system such as the one presented in section I of this chapter. Suppose it is desired to have the system output displacement go from position A to position B in the minimum amount of time. If there are no physical constraints on the system, the fastest response could be achieved by first accelerating the output to infinite velocity in zero time, then moving from point A to a position corresponding to point B at this infinite velocity, then decelerating to zero velocity in zero time. This method is represented by the phase portrait in Figure 2.5, where x is the output displacement and v is its first derivative, the output velocity. It may be noted in passing that the area under the phase

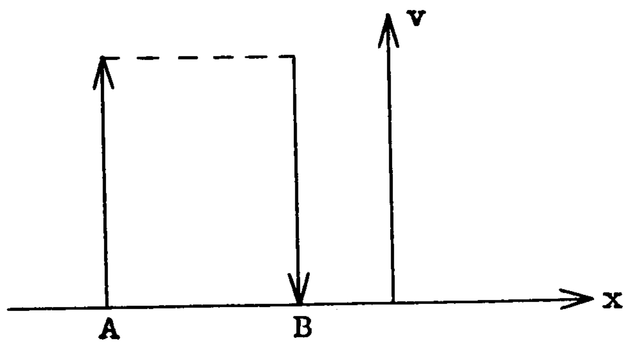

Figure 2.5. Phase Portrait with no System Constraints.

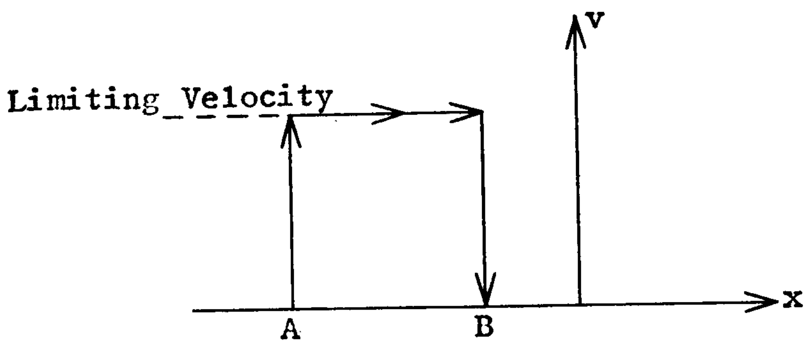

Figure 2.6. Phase Portrait with Velocity Constraint.

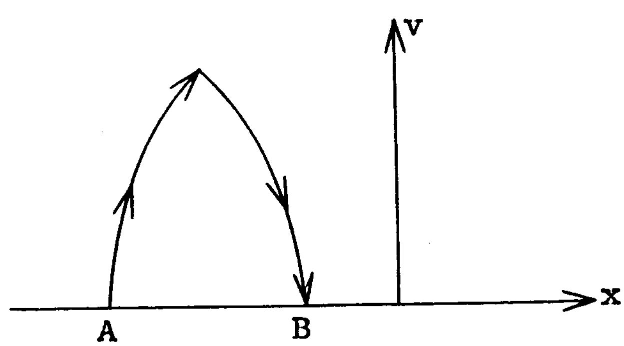

Figure 2.7. Phase Portrait with Acceleration Constraint.

trajectory shown in Figure 2.5 is infinite, corresponding to zero elapsed time.

A slightly more practical but still unrealizable method of moving the output from A to B uses velocity-limiting. If the velocity of the system output is limited to some maximum finite value, the phase trajectory shown in Figure 2.6 results. Although the time required to go from A to B is finite, as seen from the finite area under the trajectory in Figure 2.6, infinite acceleration and deceleration forces are still required.

A more practical arrangement assumes the limitation on acceleration. Even this limitation is still somewhat unrealistic, since it assumes that the accelerating force is constant throughout the velocity range, and is instantly reversible. However, the type of trajectory shown in Figure 2.7, including the instantaneous change in acceleration, can be obtained from the system being considered here, since it is ideal and second-order. If it were possible to maintain a constant level of acceleration throughout the trajectory, it has been shown[8] that each half of the trajectory would be parabolic. If the system being studied here were of third order or higher, a limitation would be placed on the rate of change in acceleration, and the sharp change indicated in Figure 2.7 would

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[8] Gibson, *op. cit.*, p. 440.

not be realizable. The shape of the trajectory of Figure 2.7 suggests that the quickest way to get to point B from point A involves switching from maximum accelerating to maximum decelerating effort midway in the trajectory. The mathematical justification for this "full-forward, full-reverse" method of switching was made earlier in this chapter, when the "bang-bang" solution was presented.

Choice of D. R. C. System

The primary purpose of this study is to compare the time response of a discontinuous rate compensated system with that of a bang-bang configuration of the same system. The bang-bang system was chosen for study because it is widely accepted and has been extensively studied. However, the bang-bang solution has major physical drawbacks. First, the drive on the plant must be turned off at the instant the output variables reach their final state. If this turn-off is not accomplished at the right time, or at all, limit cycle operation may result. Second, the switching point must be precicely set so that the final output state may be reached in one switching.

The d. r. c. system was chosen for comparison with the bang-bang system. This choice was made because the d. r. c. configuration is not physically limited by the disadvantages inherent in the bang-bang system. In a

d. r. c. configuration, the steady-state output error is zero, so the final state will always be reached by the output variables. Furthermore, this state will be maintained by the system without it being necessary to turn off the drive to the plant. Also, the switching point in a d. r. c. system is not critical. The output will reach its final state regardless of the switching point. In fact, variation of the switching point in a d. r. c. configuration can be used to vary the system output response until "optimum" response, with respect to some preselected criteria, is obtained. Two switching schemes were employed in the study of the time response of d. r. c. systems. In one case, the system was switched such that the output would overshoot by 5% for any level of input step. In the second case, a switching point set to some constant output level was used. The results of these studies are presented in Chapter IV.

III. CONCLUSIONS

In this chapter, it was seen that the analysis of even simple, second-order feedback control systems by conventional automatic control theory can be very tedious. Although the use of a digital computer eliminated most of the repetitive computational effort, algebraic manipulations involved in finding the exact time solutions of the systems studied remained laborious. Moreover, the conventional method of analysis was found not to lend itself readily to optimization. It was mentioned that the conventional method does not easily provide such information as maximum output velocity and acceleration.

A study of time-optimal control theory showed that, with constraints on the control effort, the value of the control function u(t) could only be +1 or -1; hence, the name "bang-bang". It was indicated that finding the time function of the control signal would be very difficult analytically, even for a second-order system, but that this function could be found fairly easily by analog computer techniques.

The phase plane concept was studied. Although this concept was found to be a useful tool, it was found to have one serious drawback; the phase plane is valid only for second-order systems. However, since the system under consideration in this thesis was of second order, this limitation on the phase plane was of no concern. The phase plane was also found to be useful in presenting and visualizing optimization concepts with regard to switched systems.

The reasons for choosing a discontinuous rate compensated system configuration for output response comparison with the bang-bang system were presented. The inherent physical drawbacks of the bang-bang configuration were brought out. It was pointed out that the d. r. c. system posesses none of these drawbacks.

In the study thus far, a strong case has been made for the use of analog computer models for finding time-optimal solutions in control systems. It will be seen how these models are constructed and used on an analog computer to provide a large amount of information on the solution to the problem at hand; that is, of comparing discontinuous rate compensation with the well-known and widely-studied bang-bang time-optimal solution.

CHAPTER III

ANALOG COMPUTER MODEL BUILDING

The need for building analog computer models of the systems under consideration was established in Chapter II. Procedures for developing and using such models, as well as for checking their validity and accuracy, will be studied in the present chapter. First, the principles of analog computation will be studied briefly to explain how an analog computer is used to represent physical systems. Next, models of the various system configurations being studied will be constructed utilizing these principles. Finally, the validity and the accuracy of the outputs of these models will be checked against the outputs obtained in Chapter II. This chapter is divided into the following sections: (1) analog computer principles and (2) development of system models.

I. ANALOG COMPUTER PRINCIPLES[1]

An analog computer is used to solve linear, constant-coefficient differential equations. The computer can also be used to solve other types of equations, but its

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[1]Basics of Analog Computer Programming Handbook, (Long Branch, New Jersey: Electronic Associates, Inc.), pp. 1-5.

application to this study is limited to this one class of equations. Forcing functions and/or initial conditions can also be simulated on an analog computer. The equation to be solved is, in effect, a mathematical model of the system it describes. This equation is solved continuously from some reference time $t = 0$, when solution is initiated by the operator. Any of the system variables can be obtained from the model while solution is in progress. These variables are represented by voltage levels at various points on the model.

The differential equation to be simulated on the analog computer must be of the correct form to be easily programmed. To be in this form, the equation should be solved for its highest-order derivative, then normalized with respect to this term. For example, consider the following differential equation:

$$a\dot{x} + bx = c \tag{3.1}$$

where $\dot{x} = dx/dt$. The correct analog form of this equation would be

$$\dot{x} = c/a - bx/a \tag{3.2}$$

The method of programming used in this study is called the "bootstrap" method. This method assumes that the normalized, highest-order term in an equation is already available, then constructs this term from itself by integration, attenuation and summation. Once the analog model of the equation has been constructed, forcing functions and/or initial conditions are added to the program. For example, if equation (3.1) were to be simulated, the programmer would first assume that the term $\dot{x}$ is available as a voltage, then he would construct this term by producing the term (c/a - bx/a) from it.

The components which are available on an analog computer and which apply to the present study, along with their mathematical uses, are the following:

1. Potentiometers--used as attenuators.
2. Operational amplifiers--used as inverters, summers, integrators, multipliers, or in combinations of these functions.
3. Electronic comparator--used as an electronic switch to produce discontinuous control signals.

For illustration, an analog model of the system represented by equation (3.1) will be constructed. Using the bootstrap method of programming, it is first assumed that a voltage proportional to $\dot{x}$ exists. This voltage is integrated with respect to time, producing a signal proportional to -x (the integrator also inverts the signal). This voltage is then multiplied by the value of b/a to produce one term in the original equation. If the magnitude of b/a is less than unity, the multiplication can be performed by means of a potentiometer. If the magnitude of b/a is between 1 and 10, the integrator can be used to provide a multiplication of 10, and then a potentiometer can be used to attenuate the signal to its proper value. If the magnitude of b/a is greater than 10, more amplifiers will be required. It is assumed for simplicity that the magnitude of b/a is less than unity.

Next, a forcing function of magnitude c/a is formed by means of a reference voltage attenuated by a potentiometer to produce c/a volts. Again, amplification may be required. It is assumed here that the reference voltage is 10 volts (as it was in the analog computer used in this study) and that the magnitude of c/a is less than 10. Now that the terms c/a and (-bx/a) have been formed, they are summed to form $\dot{x}$, the originating voltage. A practical analog computer configuration for solving equation (3.2) with the previously-stated assumptions is shown in Figure 3.1.

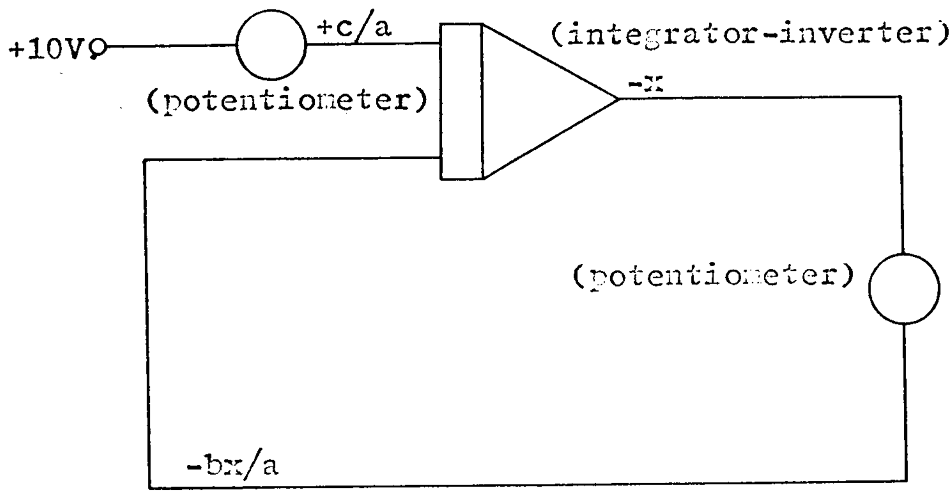

Figure 3.1. Analog computer model of $\dot{x} = c/a - bx/a$.

II. DEVELOPMENT OF SYSTEM MODELS

This section concerns the development of analog computer models representing the basic system studied in both the bang-bang configuration and the discontinuous rate compensated configuration, where the ideal relay is placed in the tachometric feedback loop. In each case, both system #1 and system #2 were developed. The configuration being evaluated in this thesis is, of course, the discontinuous rate compensated one. The bang-bang configuration, being the theoretical "time-optimal" solution subject to certain assumptions, was used as a standard for comparing the performance of the d. r. c. system.

Also included in this section are checks of the validity and accuracy of the outputs of the models developed. These checks are made by simplifying the models to the basic configurations seen in Chapter II (Figures 2.1 and 2.3). The outputs of these simplified models were compared with those obtained analytically and calculated by digital computer (Figures 2.2 and 2.4). The analog model outputs agree closely with those obtained analytically. It can be reasonably assumed, then, that the introduction of a discontinuity in either system does not invalidate the model of that system, since the systems are still piecewise linear.

Bang-Bang System Model

In Chapter II, it was seen that time-optimal response could be obtained, under certain assumptions, by means of the so-called "bang-bang" solution. The basic system being considered in this thesis, including the different configurations studied, can be represented in the general bang-bang configuration shown in Figure 3,2, where $G(s) = K_1/s(s+b)$ and $H(s) = K_2 s$.

In Figure 3.2, the element N represents the ideal reversing relay developed in Chapter II. The control signal u can assume values of +1 and -1 only, simulating saturation of the amplifier K_3. The error signal $R(s) - C(s)$ serves only to trigger the relay at the proper time, allowing the system output to reach its final value in minimum time. The system can be returned to its basic configuration as discussed in Chapter II by first short-circuiting the relay N, making the control signal equal to the error signal. The tachometric feedback loop can then be removed to reproduce the basic system by letting $K_2 = 0$. The system as shown in Figure 3.2 is then a general configuration which can easily be modified to recreate system configurations studied earlier.

The analog model of this system is determined first by obtaining the system differential equation in its proper form. From Figure 3.2, it is seen that $E(s)=NK_3$ at all times, depending on the value of C(s) only

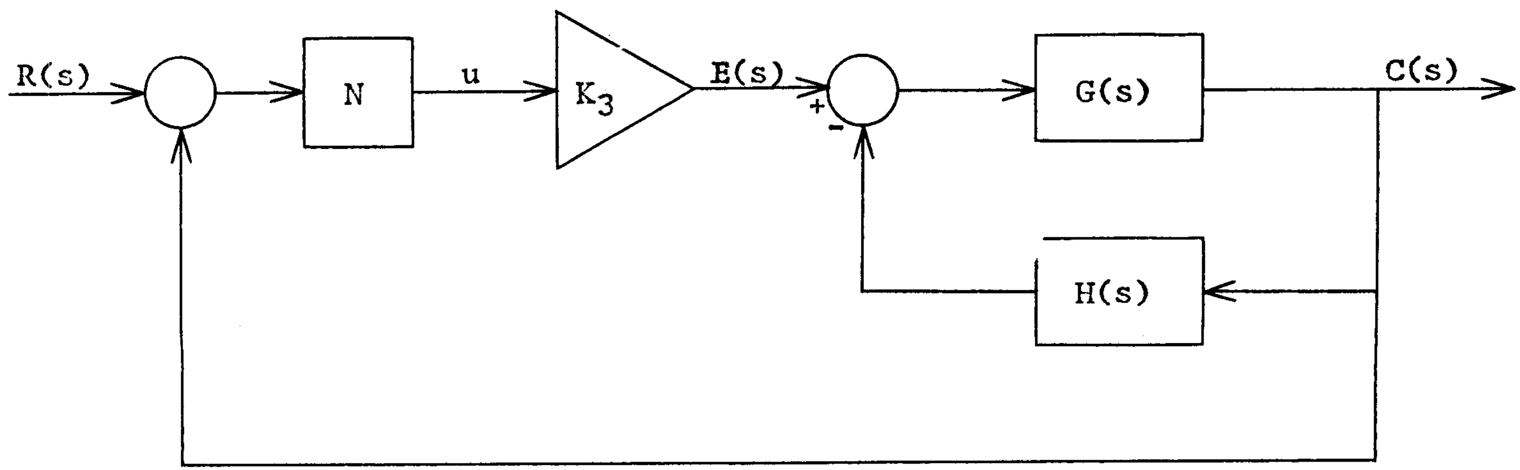

Figure 3.2. Bang-bang time-optimal system.

indirectly as a triggering criterion for the ideal relay represented by N. Therefore, it can be concluded that the large loop is not a feedback loop in the physical sense of the word. That is, on a conceptual basis, Figure 3.2 could be represented by the system shown in Figure 3.3; this representation is valid only if it is remembered that $E(s)$ does depend indirectly on $C(s)$, but that its numerical value is not a simple mathematical function of the output. The system differential equation, in the "bootstrap" form, is found as follows:

$$\frac{C(s)}{E(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K_1}{s^2+(b+K_1K_2)s} \tag{3.3}$$

$$Cs^2 + (b+K_1K_2)Cs = K_1E = NK_1K_3 \tag{3.4}$$

Taking the inverse Laplace transform of equation (3.4),

$$c + (b+K_1K_2)\dot{c} = NK_1K_3 \tag{3.5}$$

$$c = -(b+K_1K_2)\dot{c} + NK_1K_3 \tag{3.6}$$

In equation (3.6), $N = u(t)$ the control function, where $u(t) = +1$ or -1 only.

Equation (3.6) is the system differential equation in the proper form for analog computer programming. Here, $\dot{c} = dc(t)/dt$ and $c = d\dot{c}(t)/dt = d^2c(t)/dt^2$. Using the components available, a simplified analog model was constructed as shown in Figure 3.4. The block labeled "NK_1K_3" in Figure 3.4 consists of an electronic comparator

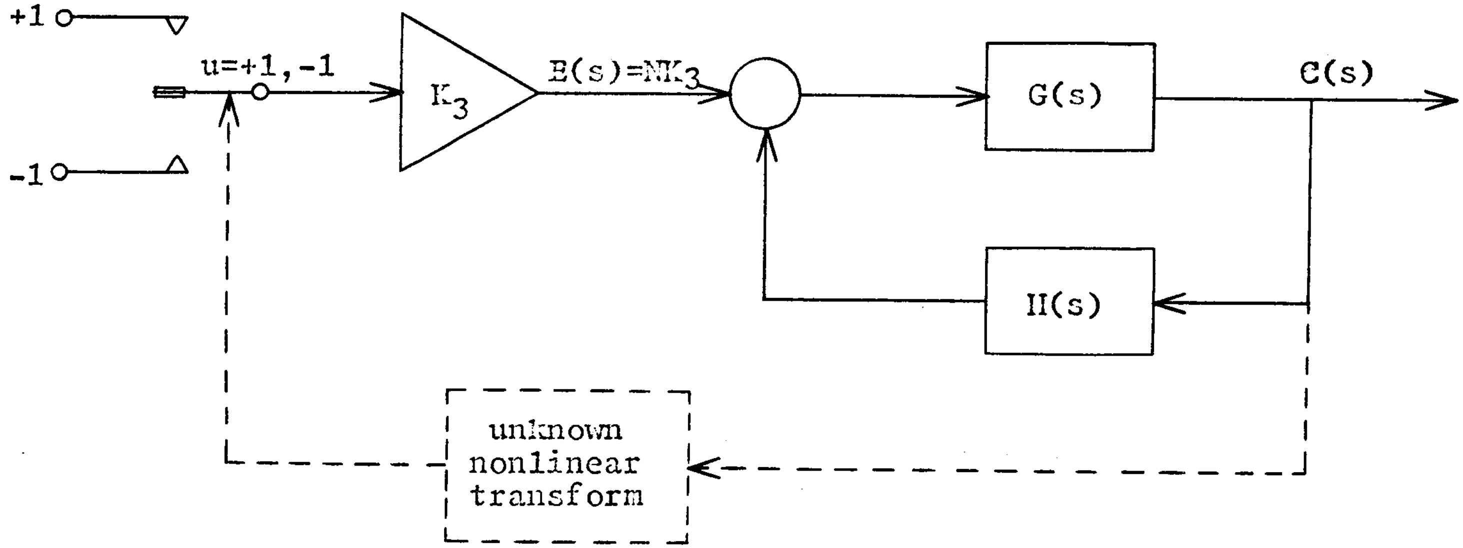

Figure 3.3. Functional bang-bang system configuration.

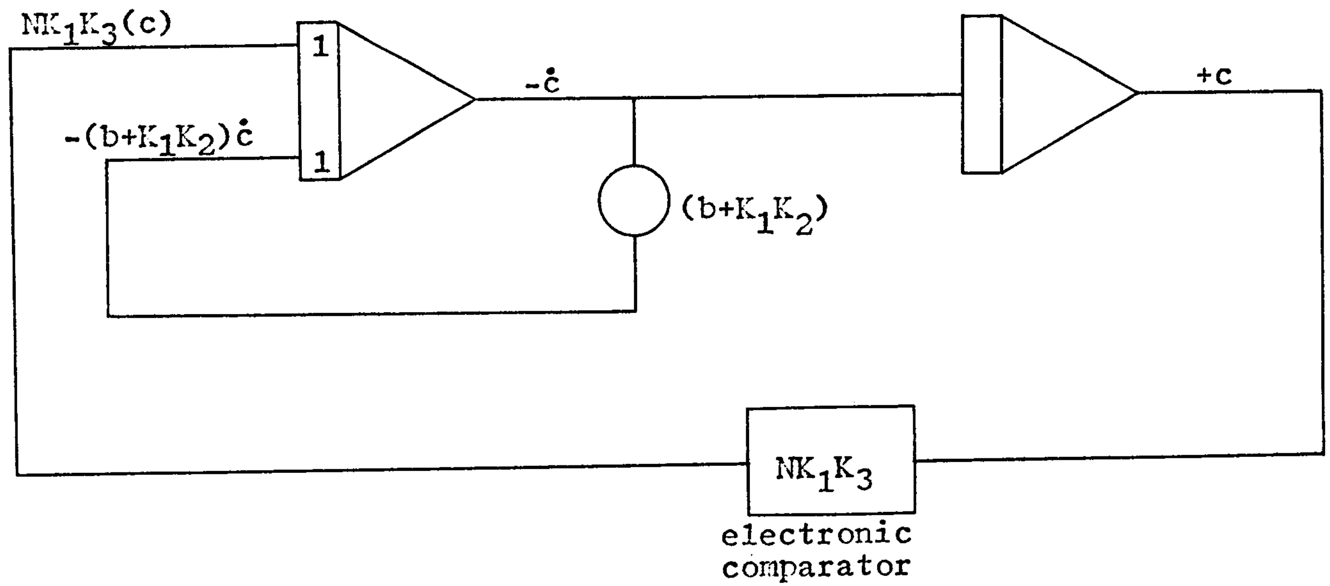

Figure 3.4. Basic analog computer model of bang-bang system.

whose output consists of $+K_1K_3$ or $-K_1K_3$, depending on the level of output c(t) needed to trigger the control signal u(t) either way. Due to physical limitations of the actual analog computer used, several trials and adjustments were necessary before a practical computer configuration was constructed as shown in Figure 3.5. The computer used was an Electronic Associates, Inc. Model TR-20.

A brief explanation of Figure 3.5 is in order. The numbers inside the potentiometer symbols (circles) represent the numbered potentiometers on the computer; the settings of these potentiometers for the various configurations studied are summarized in Table 3.1. Similarly, the encircled numbers at the head of the amplifier symbols (triangles) are the actual numbered amplifiers used. The numbers within the amplifier symbols represent the multiplication factors used. The plus and minus ten-volt levels shown are reference voltages available on the computer used. Procedures for programming the electronic comparator can be found in the computer handbook[2]. The comparator was programmed to switch the forcing function NK_1K_3 from u = -1 to u = +1 at the desired level of output c. This switching level was set by means of

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[2] The TR-20 Computer Operator's Reference Handbook, (Long Branch, New Jersey, Electronic Associates, Inc.), pp. 47-53, AII-15-16.

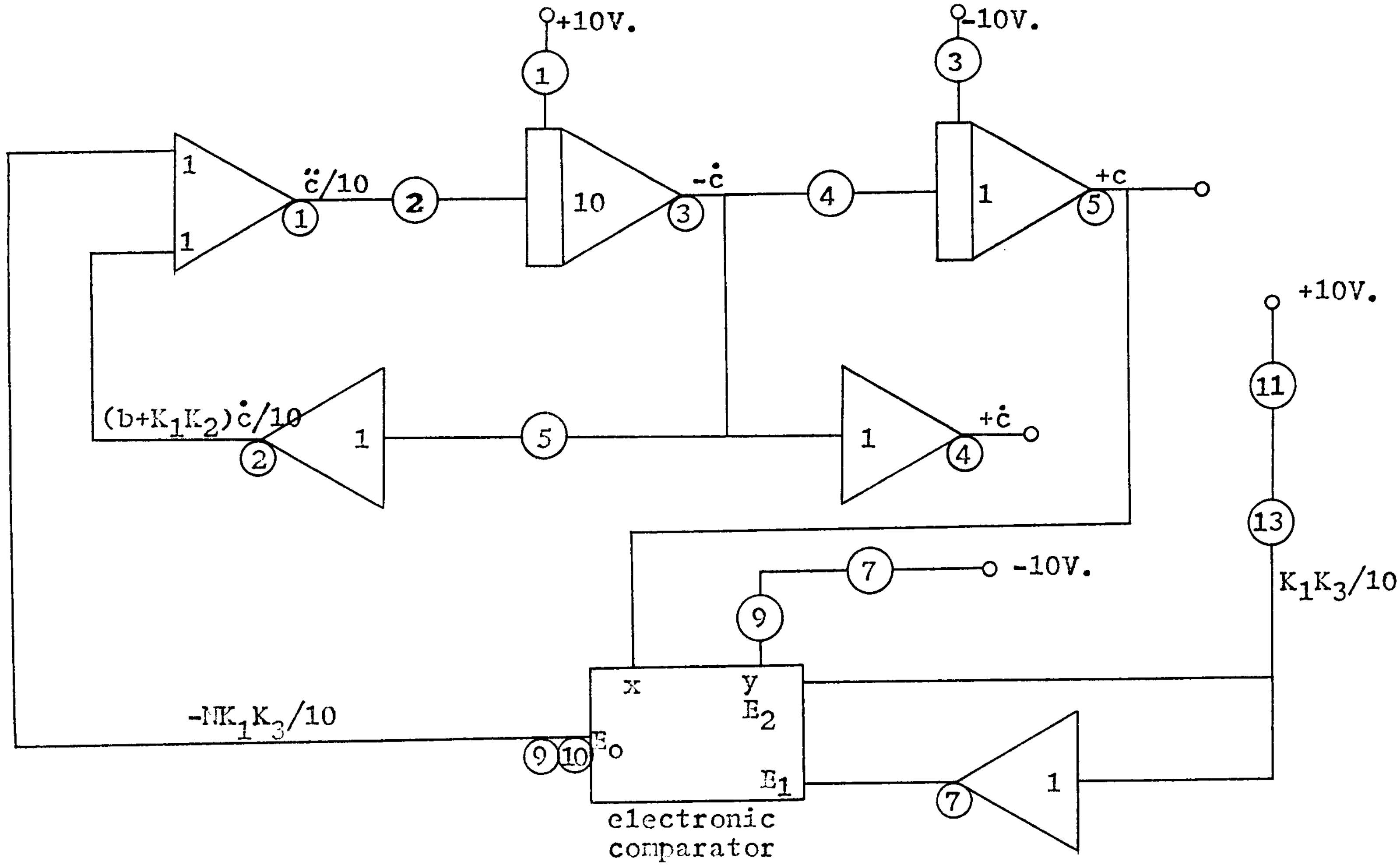


Figure 3.5. Actual analog computer model of bang-bang system.

TABLE 3.1

BANG-BANG ANALOG COMPUTER MODEL
POTENTIOMETER SETTINGS

POT. #	ASSIGNMENT	S#1 w/o T[1]	S#2 w/o T[2]	S#1 w/ T[3]	S#2 w/ T[4]
1	$\dot{c}(0)/10$	variable	variable	variable	variable
2	time scale	0.1	0.1	0.1	0.1
3	$c(0)/10$	variable	variable	variable	variable
4	time scale	0.1	0.1	0.1	0.1
5	$(b+K_1K_2)/10$	0.2	0.3	0.6	0.7
7	attenuator	0.1	0.1	0.1	0.1
9	c(switch)/10	variable	variable	variable	variable
11	attenuator	0.2	0.2	0.2	0.2
13	$K_1K_3/20$	0.71	0.81	0.71	0.81

[1]System #1 without tachometer.
[2]System #2 without tachometer.
[3]System #1 with tachometer.
[4]System #2 with tachometer.

potentiometers 7 and 9.

Discontinuous Rate Compensated System Model

The primary purpose of this thesis is to present a study of discontinuous rate compensation in feedback control systems. The objective of this study is to evaluate the time response of such a system when it is subjected to step inputs. As was mentioned earlier, the performance of the d. r. c. system was compared to that of the bang-bang configuration. In a d. r. c. system, an ideal relay is placed in the tachometric feedback loop of the driven plant. Unlike the bang-bang system, this relay merely changes the sign on the tachometric feedback signal at some pre-selected value of output, output rate or output acceleration. A block diagram of the d. r. c. system configuration studied here is shown in Figure 3.6. In order to make the response comparison between the two system configurations more meaningful, the amplifier K_3 was allowed to saturate at the same level in the d. r. c. system as it did in the bang-bang system. To evaluate the effect of this saturation on system performance, studies were made with and without saturation in K_3. As before, both system #1 and system #2 parameters were used.

The analog computer model of the d. r. c. system was developed in the same manner as was the bang-bang system model. The system differential equation in "bootstrap" form was found as follows:

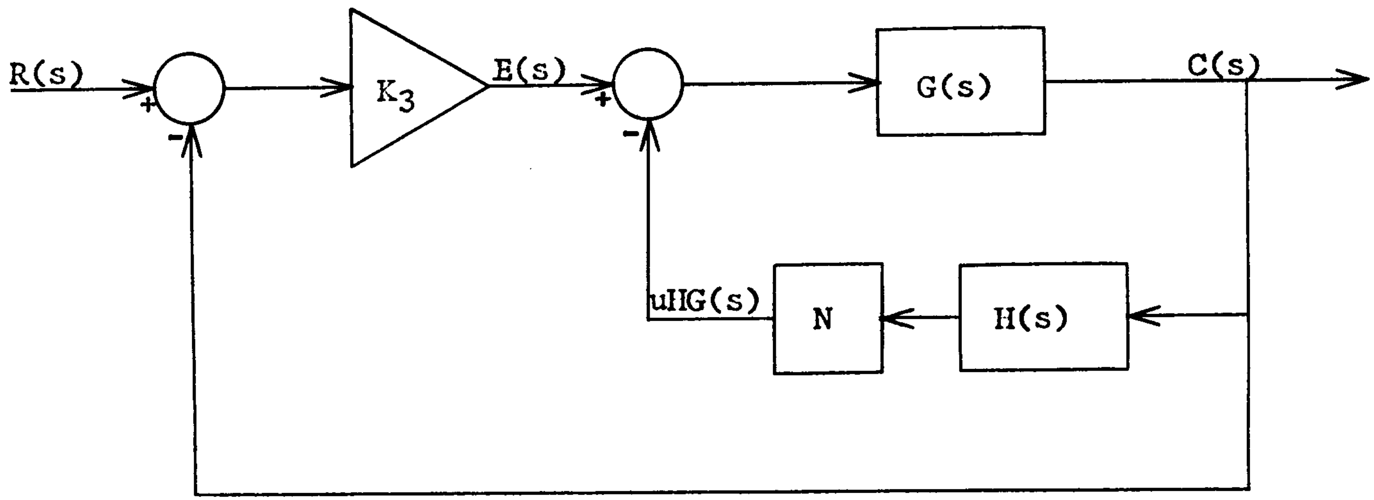


Figure 3.6. Discontinuous rate compensated system.

$$\frac{C(s)}{E(s)} = G'(s) = \frac{G(s)}{1+NG(s)H(s)} = \frac{K_1}{s^2+(b+NK_1K_2)s} \qquad (3.7)$$

$$\frac{C(s)}{R(s)} = \frac{K_3G'(s)}{1+K_3G'(s)} = \frac{K_1K_3}{s^2+(b+NK_1K_2)s+K_1K_3} \qquad (3.8)$$

$$\left[s^2 + (b+NK_1K_2)s + K_1K_3\right]C(s) = K\,KR(s) \qquad (3.9)$$

If the input r(t) is a unit step, then R(s) = 1/s. For a step input, the final value of c(t) from equation (3.8) and the final value theorem is the same as the magnitude of the input step. That is, the steady-state error in the system is zero. For this type of system, letting the input be zero and assigning some initial condition to the output is mathematically the same as letting the input be a step of the same magnitude with no initial conditions. In the first case, the output will go from some initial value to zero; in the second case, the output will go from zero to the magnitude of the step. The mathematics and programming are simplified by letting R = 0 and assigning an initial value to C. Taking the inverse Laplace transform of equation (3.9), letting R = 0 and solving for the highest-order derivative yields

$$c = -(b+NK_1K_2)\dot{c} - K_1K_3c \qquad (3.10)$$

The analog computer model of the d. r. c. system, represented by equation (3.10), was constructed by the same method used to construct the model of the bang-bang system. Again, the electronic comparator was used as the ideal relay. The actual model used for the d. r. c. system is shown in Figure 3.7. The block labeled "saturation" in Figure 3.7 is that part of the model which simulated saturation in amplifier K_3. The program of the "saturation" block is shown in Figure 3.8. This block was connected in such a way that it could either be included in or excluded from the model, so that the effects of saturation on the system response could be studied. The comments explaining Figure 3.5 in the previous sub-section of this chapter also apply to Figure 3.7. The settings of the potentiometers for the various configurations studied are summarized in Table 3.2.

Validity and Accuracy Tests for Basic Analog Models

The validity of the basic analog computer models and the accuracy of their responses were checked. These checks were made by simplifying the models developed in the previous two sub-sections of this chapter to their basic forms. The basic forms of both models were found to be identical. This finding was correct, since both system configurations were developed from the same basic control system, shown in Figures 2.1 and 2.3. In Chapter II, analytical responses were obtained for the basic system (Figure 2.1) and for the basic system with tachometric feedback (Figure 2.3). In each case, both system #1 and system #2 parameter sets were used. The analog computer model of the basic system in each of these four

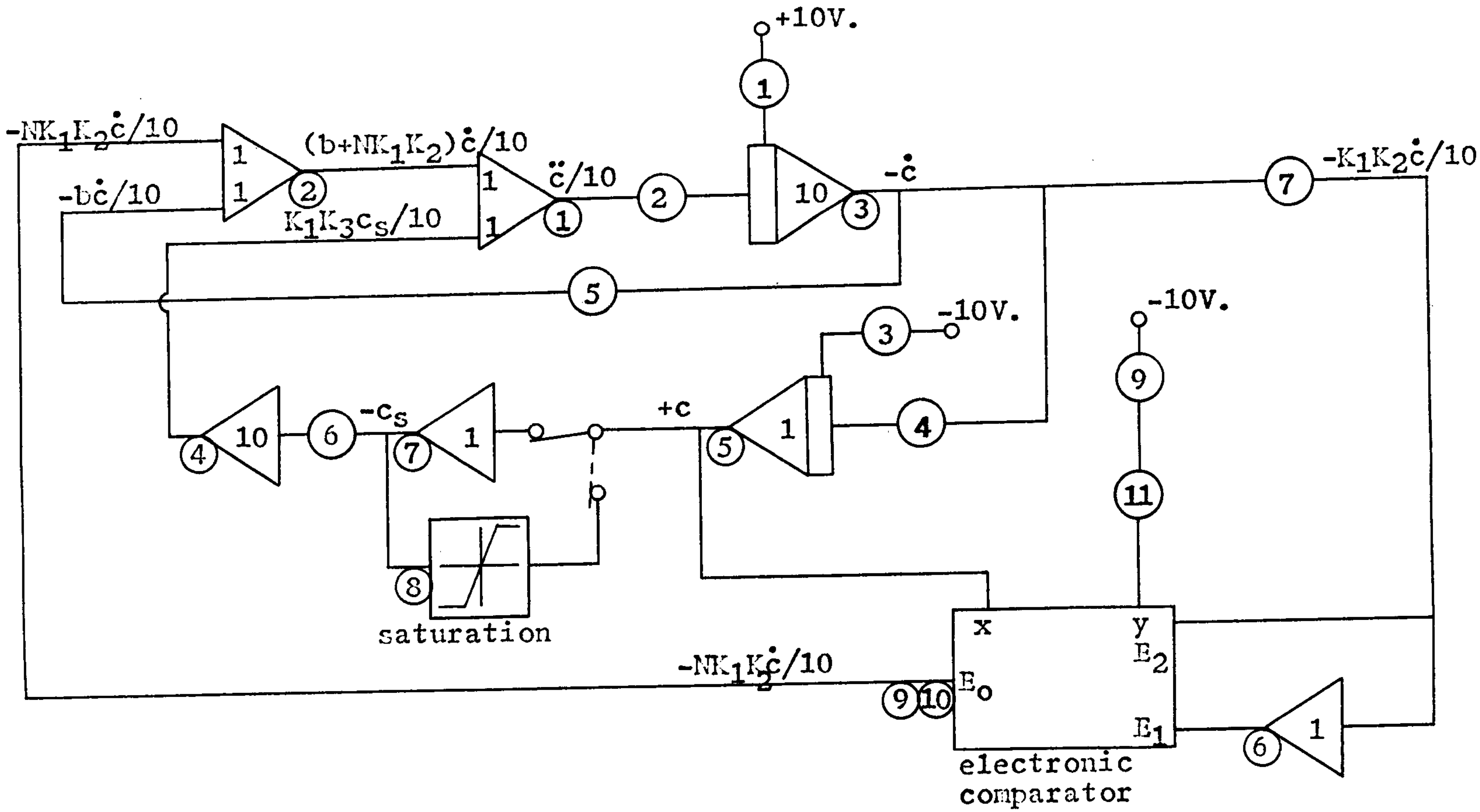


Figure 3.7. Analog computer model of d. r. c. system.

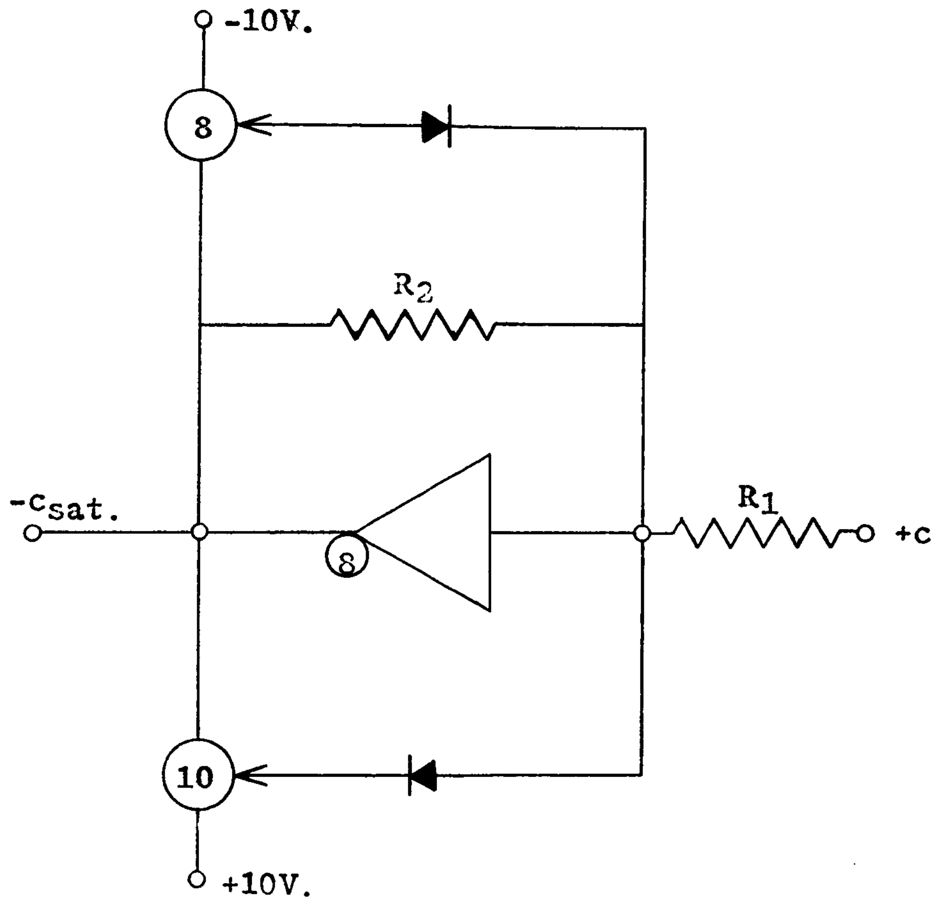


Figure 3.8. Analog simulation of saturation.

TABLE 3.2

D. R. C. SYSTEM ANALOG COMPUTER MODEL
POTENTIOMETER SETTINGS

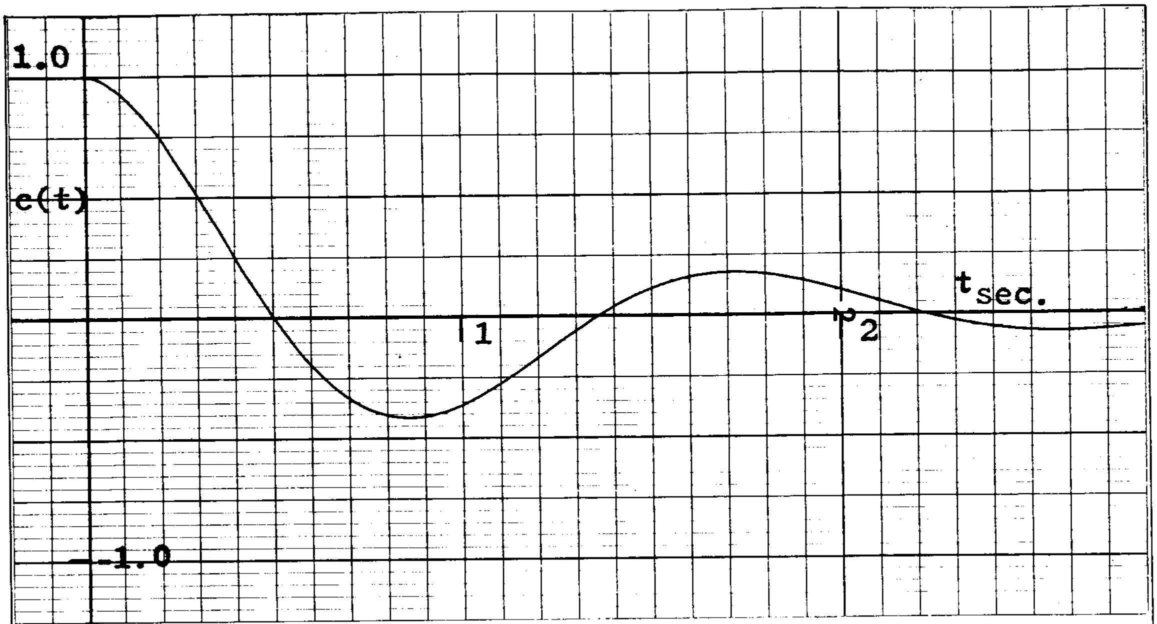
POT. #	ASSIGNMENT	S#1 w/o Sat.[1]	S#2 w/o Sat.[2]	S#1 w/ Sat.[3]	S#2 w/ Sat.[4]
1	$\dot{c}(0)/10$	variable	variable	variable	variable
2	time scale	0.1	0.1	0.1	0.1
3	$c(0)/10$	variable	variable	variable	variable
4	time scale	0.1	0.1	0.1	0.1
5	$b/10$	0.2	0.3	0.2	0.3
6	$K_1K_3/100$	0.142	0.162	0.142	0.162
7	$K_1K_2/10$	0.4	0.4	0.4	0.4
8	sat. level set.	0.05	0.05	0.05	0.05
9	attenuator	0.5	0.5	0.5	0.5
10	sat. level set.	0.05	0.05	0.05	0.05
11	c(switch)/5	variable	variable	variable	variable

[1] System #1 without saturation in K_3.
[2] System #2 without saturation in K_3.
[3] System #1 with saturation in K_3.
[4] System #2 with saturation in K_3.

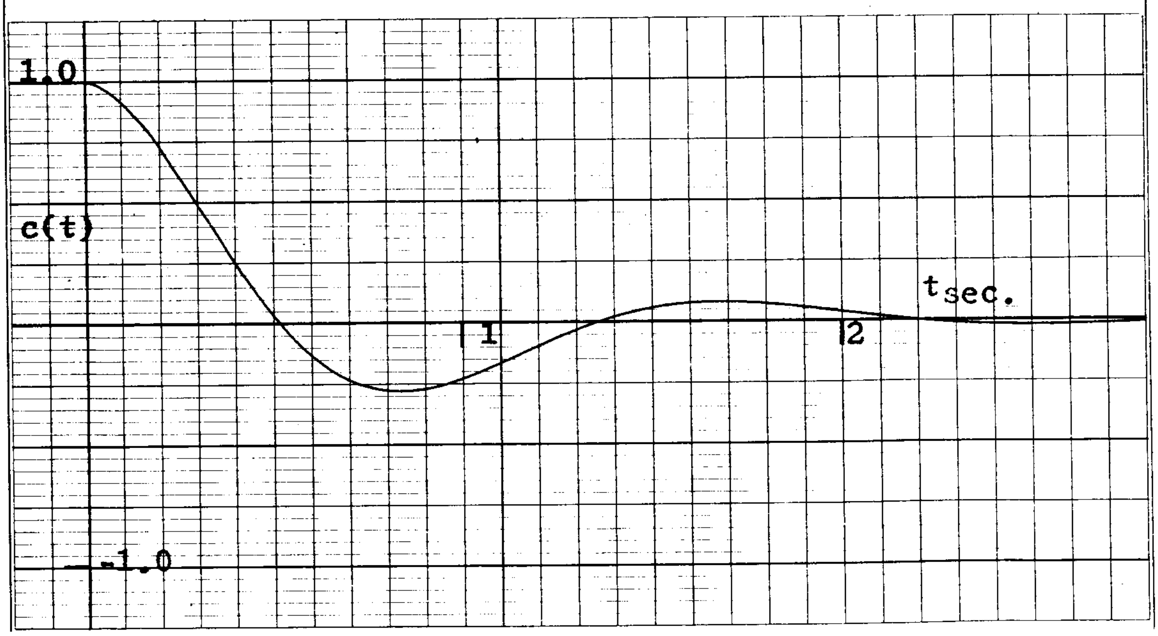
configurations was constructed. Responses of the four models to unit step inputs were obtained and were compared with the analytical responses.

Analog computer model validity was tested by comparing the output responses obtained from the analog models with those obtained analytically. The output responses to unit step inputs obtained analytically were plotted in Figures 2.2 and 2.4 for the four basic configurations; namely, system #1 and system #2 for both the basic system and the basic system with tachometric feedback. The output responses of the four basic configurations obtained from the analog computer are shown in Figures 3.9 and 3.10. In Figure 3.9, the output responses of the basic system (Figure 2.1) with system #1 and system #2 parameter sets are shown. In Figure 3.10, the responses of the basic system with tachometric feedback (Figure 2.3) with both parameter sets are shown. Figures 3.9 and 3.10 were compared with Figures 2.2 and 2.4. The curves in Figures 3.9 and 3.10 appear to be upside down because the unit step was provided by an initial output condition with R = 0. Since the two sets of output responses were nearly identical, it was concluded that the basic models are valid and accurate representations of the four system configurations in question.

The validity test was extended to the bang-bang and d. r. c. models used. The only difference between the

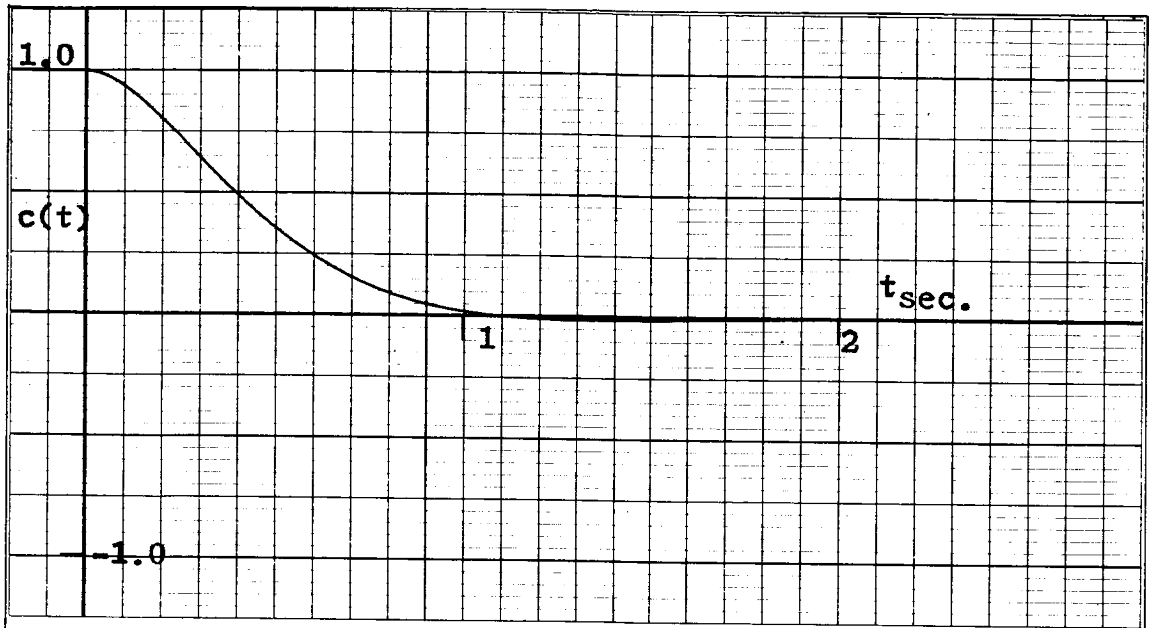


(a) system #1

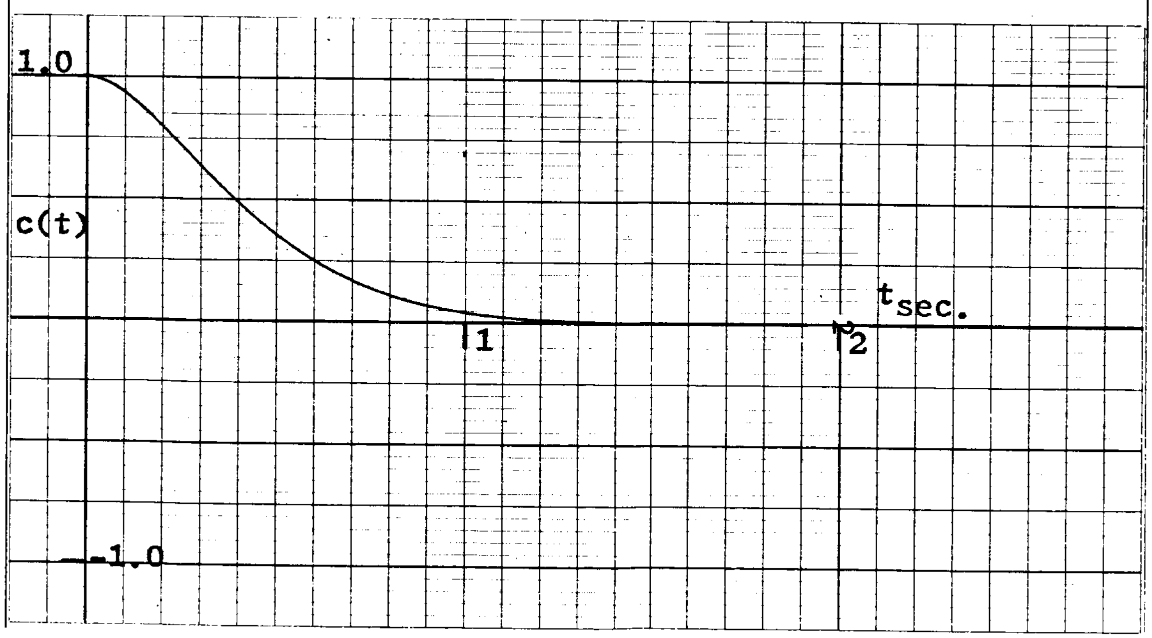


(b) system #2

Figure 3.9. Output response of basic system.



(a) system #1



(b) system #2

Figure 3.10. Output response of basic system with tachometric feedback.

bang-bang system and the basic system was the use of an ideal relay in the forward loop of the former. In both cases, the tachometric feedback could be either included or excluded. The differences between the d. r. c. system and the basic system with tachometric feedback were the ideal relay and the possibility of saturation in the d. r. c. system. The output of the electronic comparator was checked and the comparator was found to be, for all practical purposes, a perfect representation of an ideal relay.

The analog simulation of saturation diagrammed in Figure 3.8 was also checked. The voltage characteristic of the saturation simulation was obtained through the use of a unit ramp input to the circuit. The saturation characteristic was plotted by a strip-chart recorder, and this recording is included in Figure 3.11. As is obvious from Figure 3.11, the simulation was not of an ideal, "square-cornered" saturation. However, since no practical saturable element exhibits an ideal characteristic, the simulation circuit used in this study probably more nearly approximates an actual saturation characteristic than an ideal simulation would.

The bang-bang and d. r. c. systems studied, though nonlinear in the strict sense, were piecewise linear except for the saturation in the d. r. c. system. It can be logically concluded, then, that the addition of an

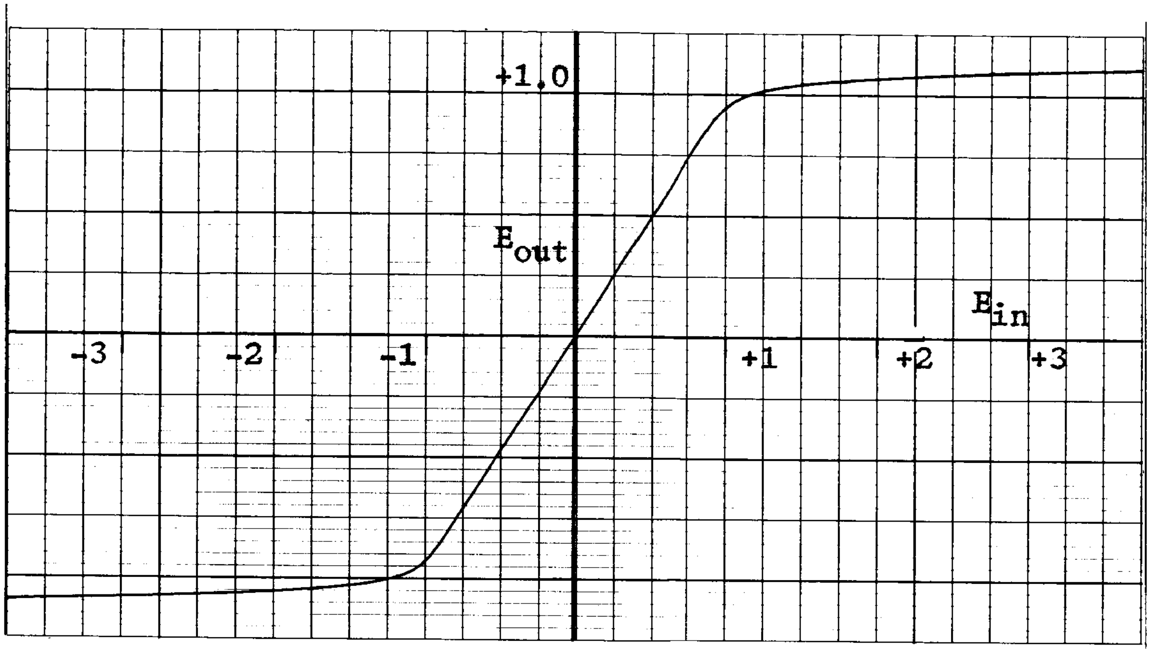


FIGURE 3.11. Voltage Characteristic of Saturation Simulator.

ideal relay to the basic system in either the forward loop or in the tachometric feedback loop has no effect on the validity of the analog models of these configurations. Also, since the voltage characteristic of the saturation simulation was known, then there is no reason to question the ability of this circuit to simulate a saturable amplifier with a similar characteristic. The analog computer models in Figures 3.5 and 3.7 can then be assumed to be good representations of the bang-bang and d. r. c. systems studied in this thesis.

CHAPTER IV

SIMULATION RESULTS AND DATA ANALYSIS

The purpose of the present chapter is to present the results obtained from the analog computer simulation studies and to provide an analysis of these results. The data summarizing these results is presented in graphical and tabular form for ease of interpretation and reference. The results are then analyzed and interpreted to evaluate the responses of the various system configurations. Although the mass of data presented was not all absolutely essential to a study of the present thesis topic in its most limited sense, it was obtained and is presented in the interests of providing aid to future studies and completeness to the present study. This chapter is divided into two sections. The first section is a presentation of the data obtained in this study. The second section is an analysis and interpretation of this data.

I. ANALOG SIMULATION DATA

Studies of the time responses of the systems for which models were constructed in Chapter III were made with an analog computer. The primary purpose of this study was to compare the time responses of the discontinuous rate compensated, or d. r. c., system with those of the time-optimal, or bang-bang, system. However, for completeness as well as for comparison, time response studies of the basic, unswitched systems were also made.

Four configurations were studied. Output response data was obtained for each of four system configurations. These configurations were the following: (1) the basic, unswitched system without tachometric feedback, (2) the basic unswitched system with tachometric feedback, (3) the bang-bang system, and (4) the d. r. c. system. Each of these four studies was made for both system #1 and system #2 parameter sets. For each study with each parameter set, outputs were obtained for the system with and without a saturable amplifier except, of course, for the bang-bang system where the amplifier is assumed always saturated. For the bang-bang system, outputs were obtained for the system without and with tachometric feedback. A total of twenty sets of outputs, or runs, were obtained; four for each configuration studied except for the d. r. c. system. In the case of the discontinuous rate compensated system, two complete studies were made; one with an "optimum" switching level such that the overshoot was a nominal 5% and one with a switching level set at some constant value of output.

The numerical and graphical data obtained from the analog computer studies is summarized and presented in this section. Since the time-response curves produced by the strip-chart recorder were so numerous, they are not included in this presentation. However, the numerical information provided by those curves was summarized in a series of tables. The phase portrait of each run was produced by an x-y plotter. These phase portraits are presented as figures in this section.

The organization of the data obtained from the studies is a series of tabulations supplemented by a series of phase portraits, each consisting of a family of response trajectories. The tables were made up in terms of response criteria studied and were divided into two groups; one for system #1 and one for system #2. The response criteria summarized into tables were the following: (1) rise time, (2) percent overshoot, (3) maximum output velocity, and (4) maximum output acceleration. The two sets of four tables are presented on the following eight pages. A total of twenty-four phase portraits are presented in the twenty figures of this section. Each simulation run provided one phase portrait except for the four d. r. c. system simulations with 5% overshoot, where switching-curve portraits were also obtained. An analysis of this data is presented in the next section.

TABLE 4.1

SYSTEM #1 RISE TIMES (IN SECONDS) FOR VARIOUS INITIAL OUTPUT LEVELS

c(0)	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	0.48	0.48	0.88	0.88	0.33	0.36	0.58	0.58	--	--
1.0	0.48	0.49	0.88	0.88	0.47	0.57	0.58	0.58	0.62	0.62
1.5	0.48	0.55	0.88	0.98	0.58	0.77	0.58	0.59	0.41	0.46
2.0	0.48	0.62	0.88	1.12	0.68	0.97	0.58	0.62	0.37	0.47
2.5	0.48	0.70	0.88	1.27	0.78	1.17	0.58	0.65	0.36	0.49
3.0	0.48	0.77	0.88	1.42	0.86	1.36	0.58	0.68	0.35	0.51
3.5	0.48	0.84	0.88	1.57	0.95	1.56	0.58	0.70	--	0.54
4.0	0.48	0.91	0.88	1.73	1.02	1.77	0.58	0.73	--	--

[1]Switching points set for constant 5% overshoot.
[2]Switching point set for constant output value.

TABLE 4.2

SYSTEM #2 RISE TIMES (IN SECONDS) FOR
VARIOUS INITIAL OUTPUT LEVELS

c(0)	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	0.50	0.50	0.93	0.93	0.31	0.35	0.50	0.49	--	--
1.0	0.50	0.50	0.93	0.93	0.45	0.56	0.50	0.50	0.70	0.70
1.5	0.50	0.55	0.93	1.06	0.57	0.76	0.50	0.53	0.43	0.48
2.0	0.50	0.62	0.93	1.19	0.67	0.98	0.50	0.57	0.38	0.48
2.5	0.50	0.68	0.93	1.33	0.77	1.18	0.50	0.60	0.37	0.50
3.0	0.50	0.74	0.93	1.49	0.86	1.38	0.50	0.63	0.36	0.54
3.5	0.50	0.79	0.93	1.66	0.95	1.59	0.50	0.66	--	0.58
4.0	0.50	0.84	0.93	1.82	1.04	1.79	0.50	0.69	--	0.59

[1]Switching points set for constant 5% overshoot.
[2]Switching point set for constant output value.

TABLE 4.3

SYSTEM #1 OVERSHOOTS (IN PERCENT) FOR VARIOUS INITIAL OUTPUT LEVELS

c(0)	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	42	42	2.5	2.5	0	0	5	5	--	--
1.0	42	42	2.5	2.5	0	0	5	5	3	3
1.5	42	40	2.5	2.0	0	0	5	5	18	15
2.0	42	36	2.5	1.5	0	0	5	5	29	23
2.5	42	33	2.5	1.5	0	0	5	5	36	25
3.0	42	30	2.5	1.0	0	0	5	5	40	28
3.5	42	27	2.5	1.0	0	0	5	5	--	29
4.0	42	25	2.5	1.0	0	0	5	5	--	--

[1]Switching points set for constant 5% overshoot.
[2]Switching point set for constant output value.

TABLE 4.4

SYSTEM #2 OVERSHOOTS (IN PERCENT) FOR
VARIOUS INITIAL OUTPUT LEVELS

c(0)	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	29	29	1	0	0	0	5	5	--	--
1.0	29	29	1	0	0	0	5	5	<1	<1
1.5	29	27	1	0	0	0	5	5	10	8
2.0	29	24	1	0	0	0	5	5	19	14
2.5	29	22	1	0	0	0	5	5	25	17
3.0	29	20	1	0	0	0	5	5	28	18
3.5	29	18	1	0	0	0	5	5	--	19
4.0	29	17	1	0	0	0	5	5	--	20

[1] Switching points set for constant 5% overshoot.
[2] Switching point set for constant output value.

TABLE 4.5

SYSTEM #1 MAXIMUM OUTPUT VELOCITY MAGNITUDES (IN RAD./SEC.)
FOR VARIOUS INITIAL OUTPUT LEVELS

	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
c(0)	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	1.30	1.30	0.82	0.81	2.52	2.00	1.5	1.7	--	--
1.0	2.65	2.65	1.63	1.63	3.45	2.30	3.1	3.2	2.8	2.8
1.5	4.00	3.70	2.45	2.14	4.10	2.37	4.6	4.6	6.0	5.8
2.0	5.25	4.40	3.26	2.37	4.60	2.42	6.2	5.9	8.8	7.8
2.5	6.60	5.10	4.08	2.48	4.90	2.45	7.7	7.1	11.7	9.7
3.0	7.90	5.50	4.89	2.53	5.23	2.45	9.3	8.3	14.5	11.5
3.5	9.30	5.80	5.71	2.58	5.48	2.45	10.7	9.5	--	13.2
4.0	10.50	6.20	6.52	2.60	5.70	2.45	12.4	10.7	--	--

[1]Switching points set for constant 5% overshoot.
[2]Switching point set for constant output value.

TABLE 4.6

SYSTEM #2 MAXIMUM OUTPUT VELOCITY MAGNITUDES (IN RAD./SEC.) FOR VARIOUS INITIAL OUTPUT LEVELS

	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
c(0)	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	1.25	1.25	0.82	0.82	2.68	2.03	1.8	1.8	--	--
1.0	2.50	2.50	1.65	1.63	3.51	2.24	3.6	3.6	2.8	2.8
1.5	3.70	3.55	2.47	2.13	4.15	2.30	5.4	5.2	5.8	5.4
2.0	5.00	4.25	3.30	2.33	4.40	2.30	7.0	6.5	8.4	7.3
2.5	6.20	4.80	4.12	2.40	4.70	2.30	8.8	7.6	10.9	8.9
3.0	7.50	5.20	4.95	2.46	4.88	2.30	10.6	8.7	13.4	10.4
3.5	8.80	5.60	5.77	2.49	5.00	2.30	12.3	9.8	--	11.7
4.0	10.00	5.90	6.60	2.51	5.12	2.30	14.0	10.9	--	12.9

[1] Switching points set for constant 5% overshoot.
[2] Switching point set for constant output value.

TABLE 4.7

SYSTEM #1 MAXIMUM OUTPUT ACCELERATION MAGNITUDES
(IN RAD./SEC./SEC.) FOR VARIOUS INITIAL OUTPUT LEVELS

c(0)	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	7.1	7.1	7.1	7.1	18.8	25.8	8.2	8.4	--	--
1.0	14.2	13.5	14.2	13.5	20.6	27.4	16.5	16.5	16.8	16.5
1.5	21.3	14.9	21.3	14.9	21.7	27.6	24.5	23.0	26.5	23.0
2.0	28.4	15.2	28.4	15.2	22.8	27.8	33.0	26.0	44.0	37.5
2.5	35.5	15.5	35.5	15.5	23.4	28.0	41.5	29.0	61.0	48.0
3.0	42.6	15.7	42.6	15.7	24.1	28.0	49.0	35.0	77.5	58.5
3.5	49.7	15.9	49.7	15.9	24.3	28.0	58.0	42.0	--	68.5
4.0	56.8	16.1	56.8	16.1	24.5	28.0	66.0	48.0	--	--

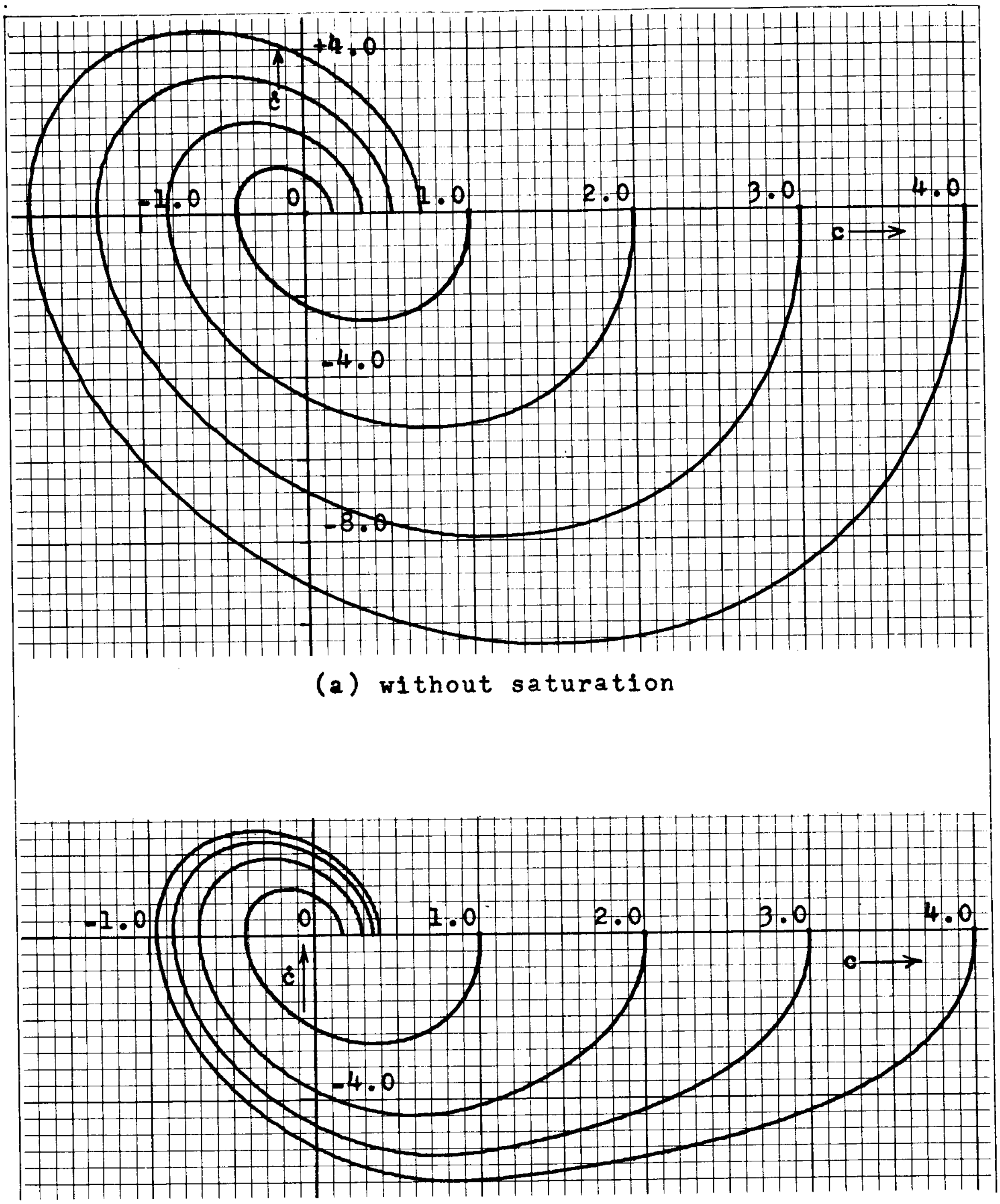
[1] Switching points set for constant 5% overshoot.
[2] Switching point set for constant output value.

TABLE 4.8

SYSTEM #2 MAXIMUM OUTPUT ACCELERATION MAGNITUDES
(IN RAD./SEC./SEC.) FOR VARIOUS INITIAL OUTPUT LEVELS

c(0)	Basic System		B. Sys.+ Tach.		Bang-Bang System		D.R.C. Sys. A[1]		D.R.C. Sys. B[2]	
	w/o sat	w/ sat	w/o sat	w/ sat	w/o tach	w/ tach	w/o sat	w/ sat	w/o sat	w/ sat
0.5	8.1	8.1	8.1	8.1	23.5	29.5	8.3	8.4	--	--
1.0	16.2	15.8	16.2	15.8	26.0	31.1	17.0	16.3	16.8	16.5
1.5	24.3	18.0	24.3	18.0	27.8	31.3	25.8	22.5	29.5	26.5
2.0	32.4	19.4	32.4	19.4	28.5	31.5	33.3	29.0	47.5	39.5
2.5	40.5	20.6	40.5	20.6	29.5	31.5	42.5	36.0	65.5	52.0
3.0	48.6	21.8	48.6	21.8	30.0	31.5	51.0	44.0	83.0	60.0
3.5	56.7	22.9	56.7	22.9	30.5	31.5	58.5	52.0	--	70.0
4.0	64.8	24.0	64.8	24.0	31.0	31.5	66.5	58.0	--	78.0

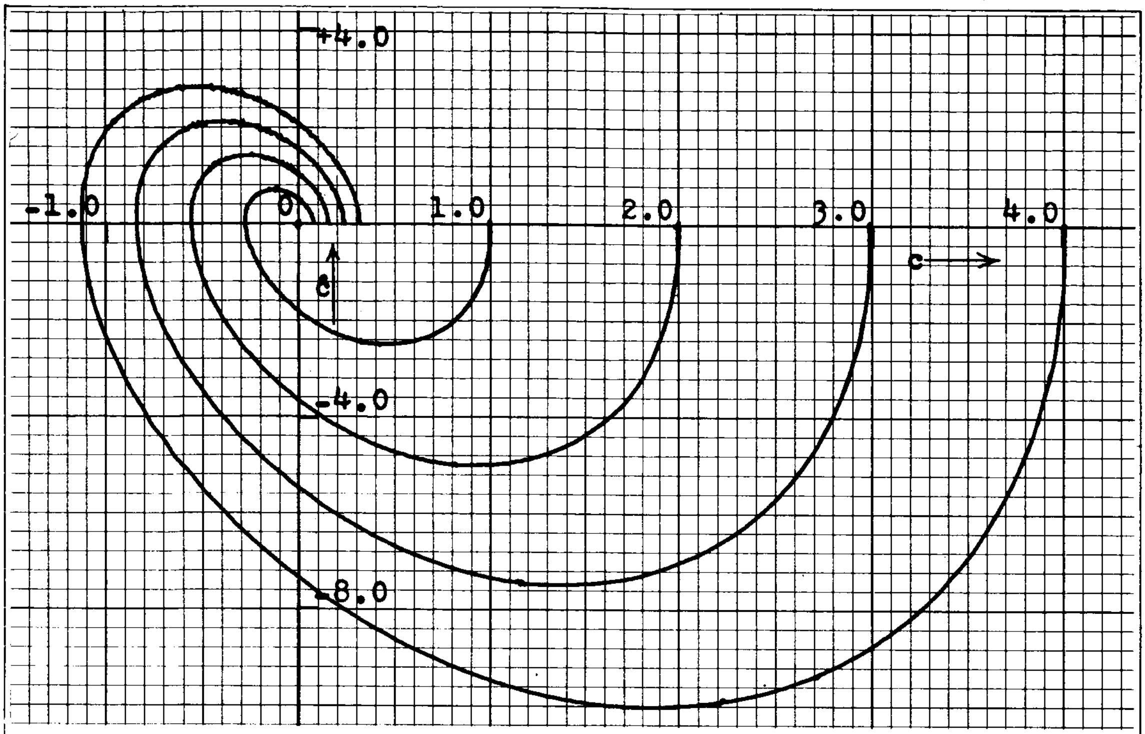
[1]Switching points set for constant 5% overshoot.
[2]Switching point set for constant output value.



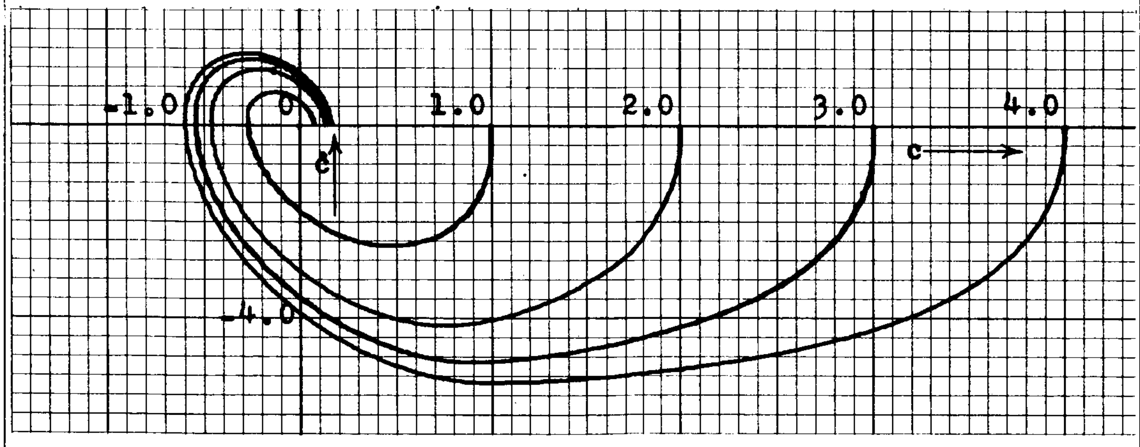
(a) without saturation

(b) with saturation

Figure 4.1. Output phase portrait of basic system #1.

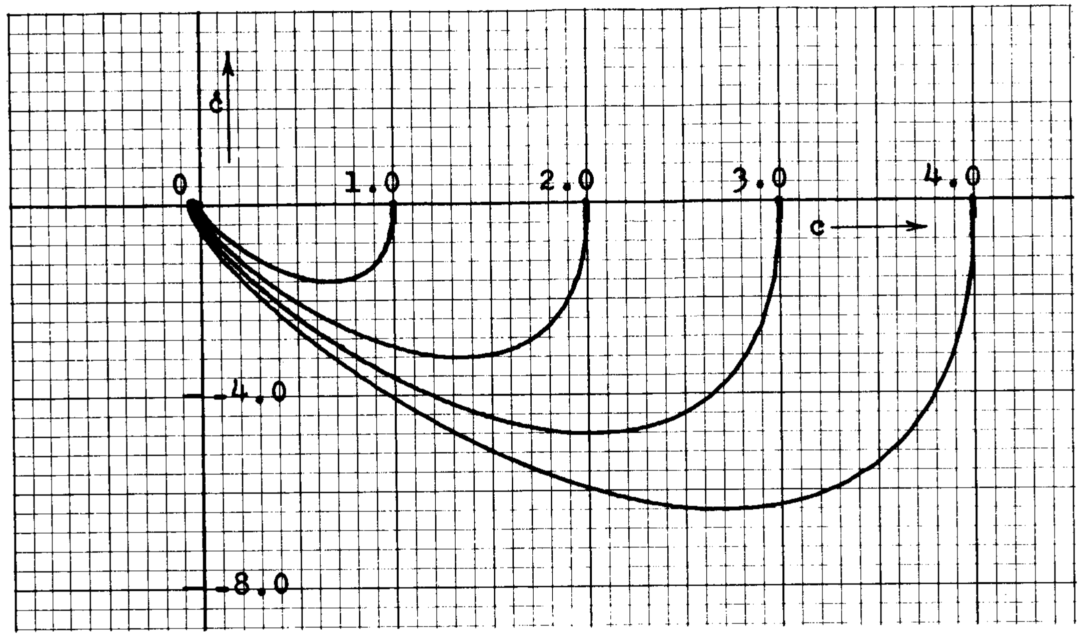


(a) without saturation

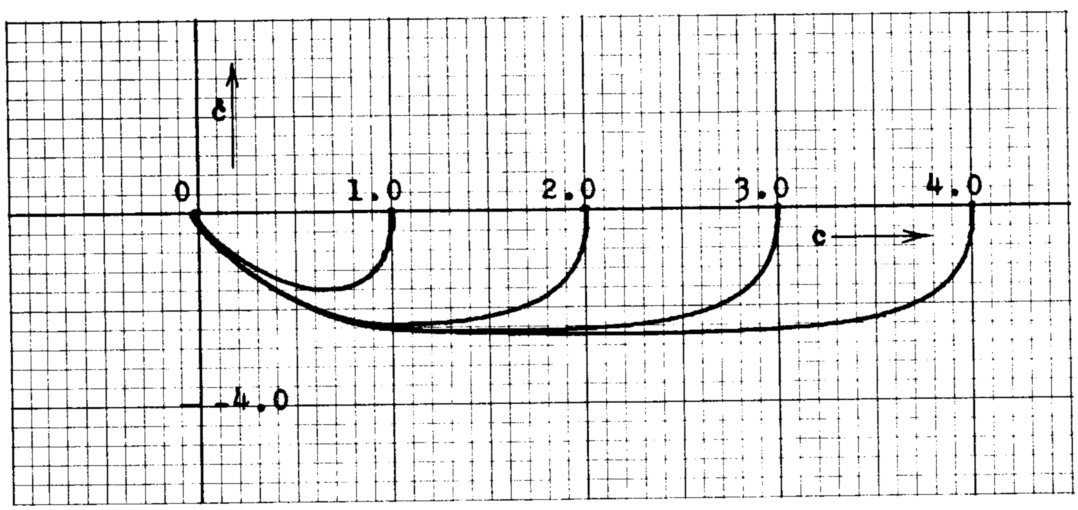


(b) with saturation

Figure 4.2. Output phase portrait of basic system #2.

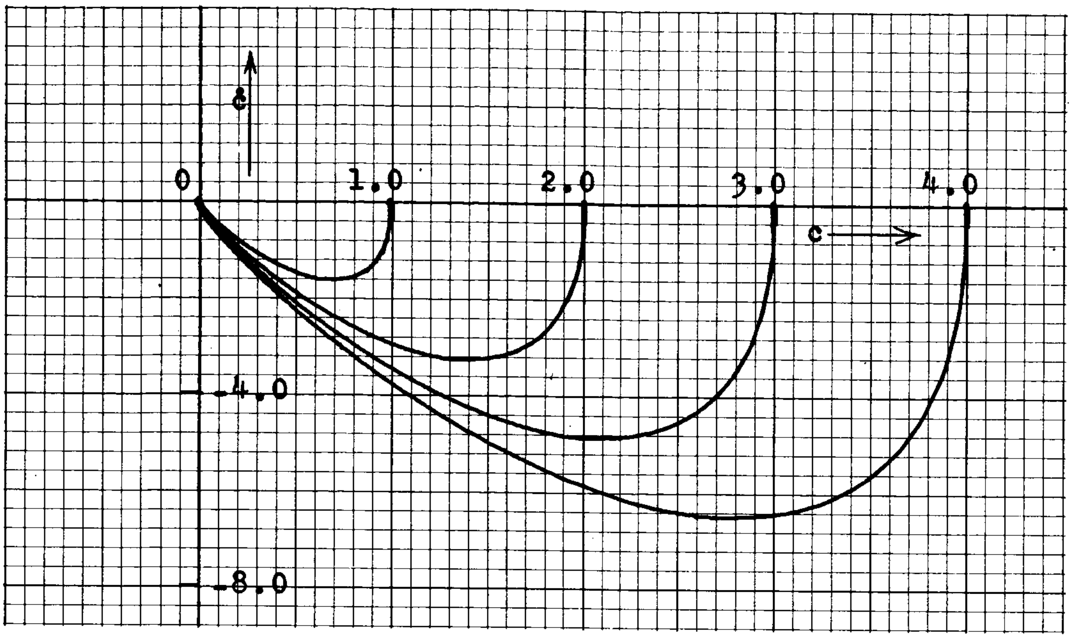


(a) without saturation

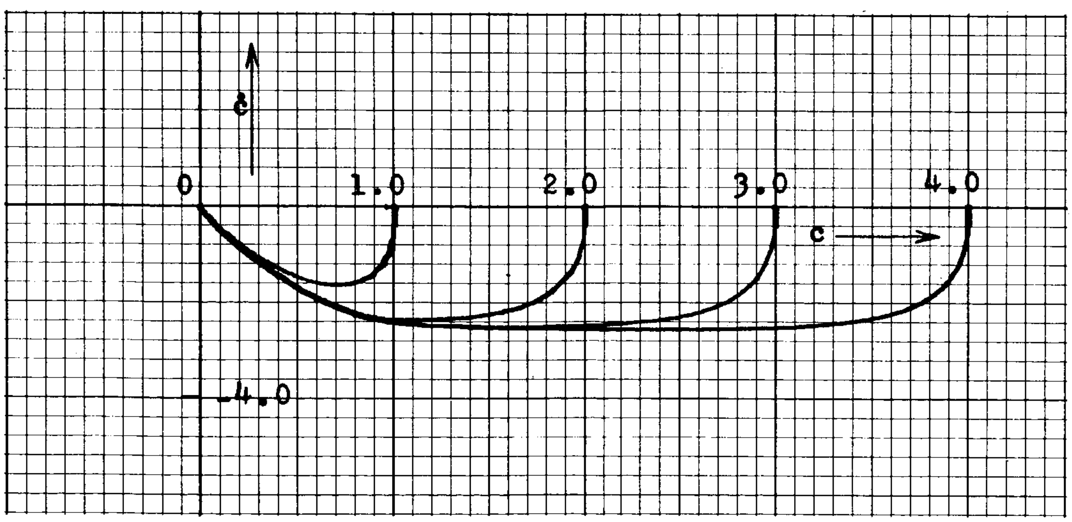


(b) with saturation

Figure 4.3. Output phase portrait of basic system #1 with tachometric feedback.



(a) without saturation



(b) with saturation

Figure 4.4. Output phase portrait of basic system #2 with tachometric feedback.

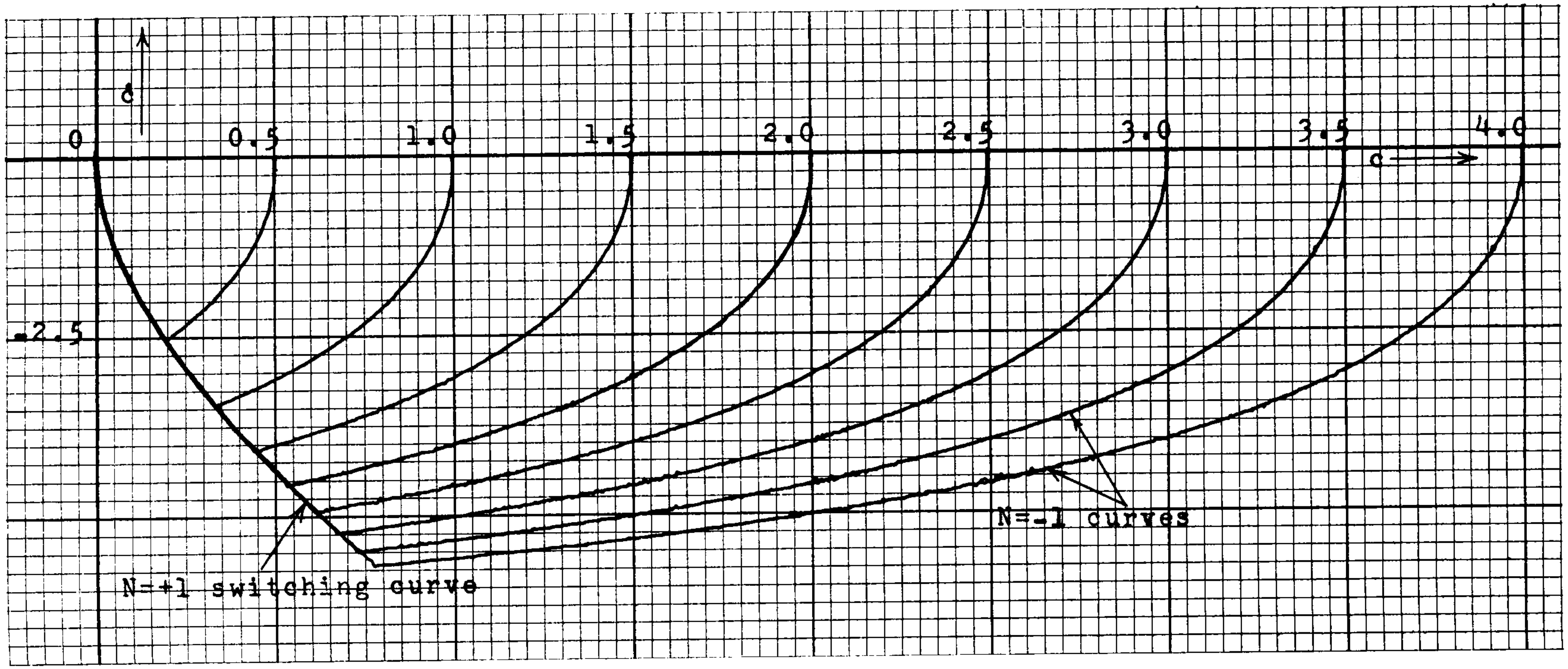


Figure 4.5. Output phase portrait of bang-bang system #1 without tachometric feedback.

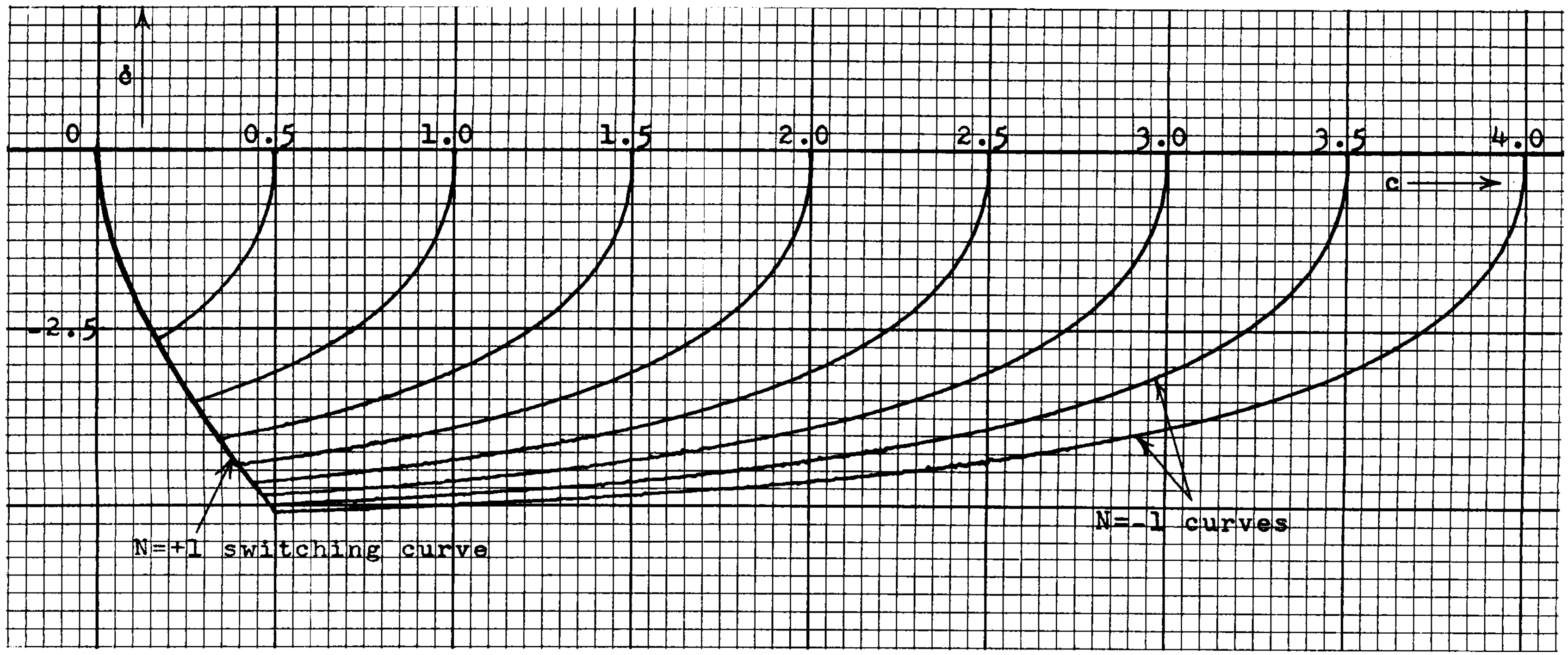


Figure 4.6. Output phase portrait of bang-bang system #2 without tachometric feedback.

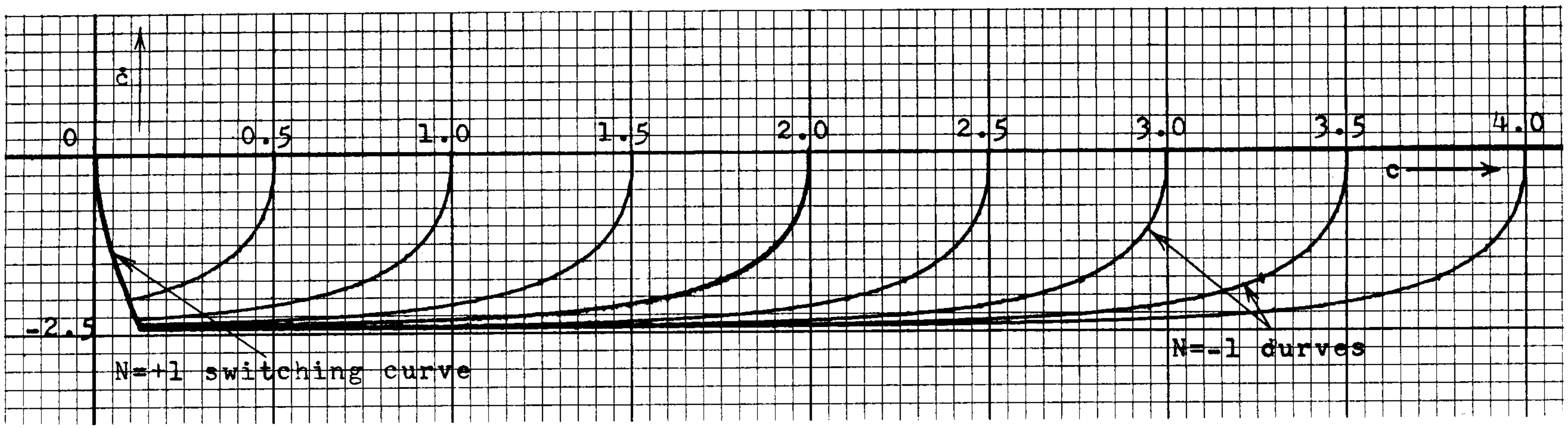


Figure 4.7. Output phase portrait of bang-bang system #1 with tachometric feedback.

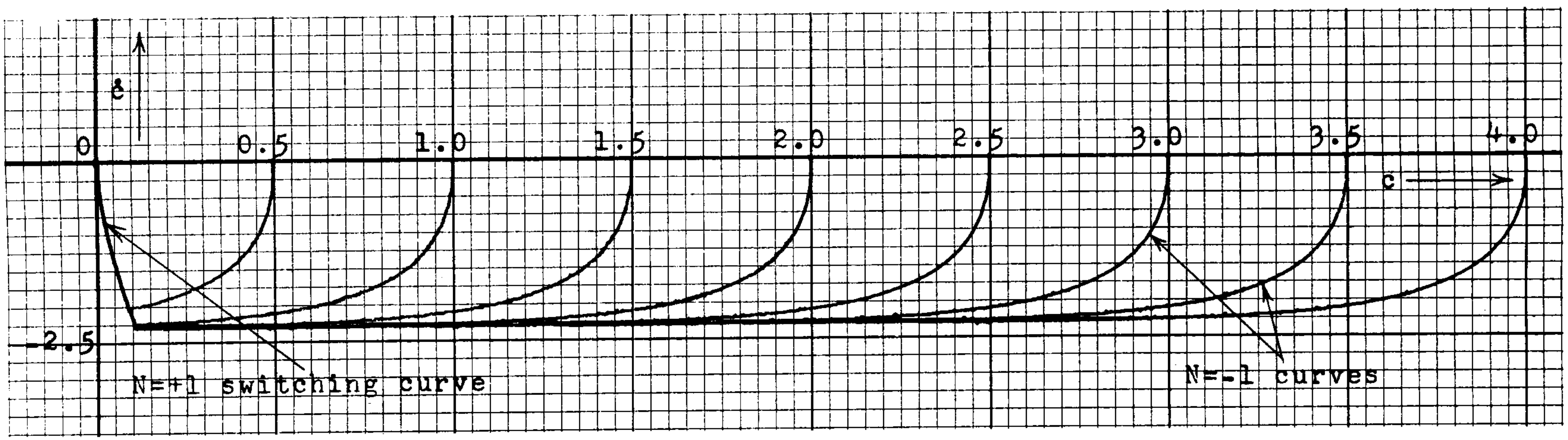


Figure 4.8. Output phase portrait of bang-bang system #2 with tachometric feedback.

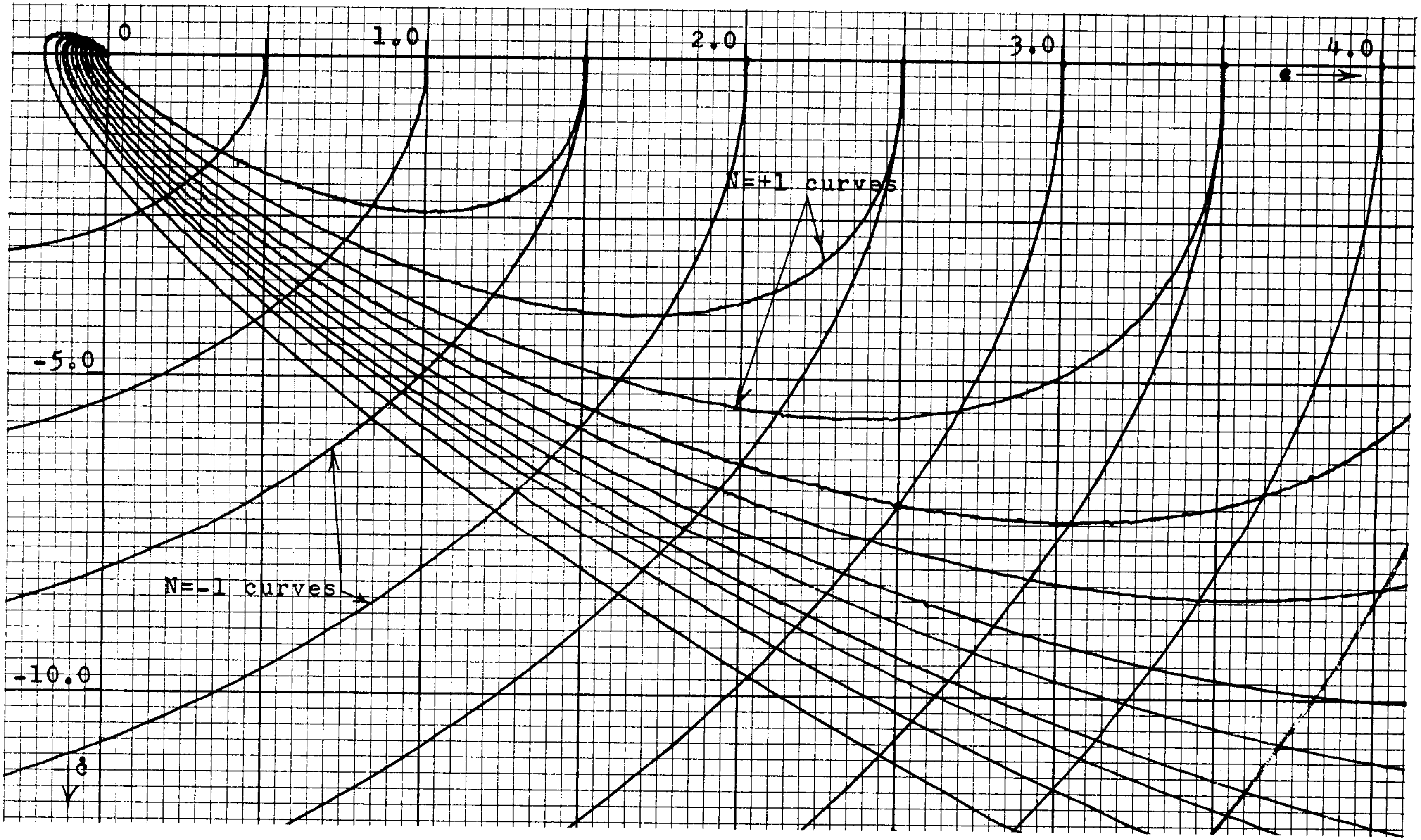


Figure 4.9. Output phase portrait of d. r. c. system #1 without saturation and without switching for N=+1 and N=-1.

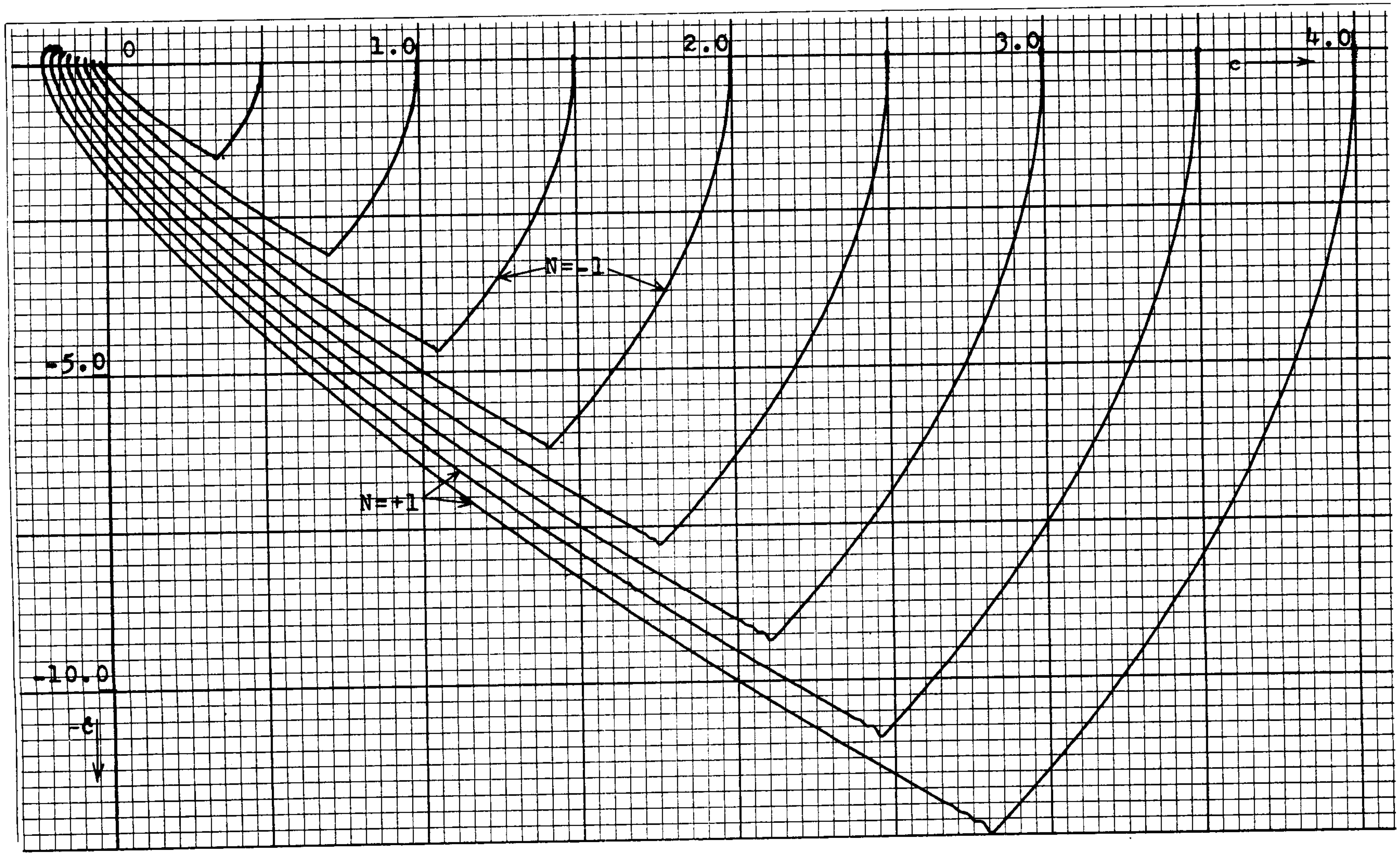


Figure 4.10. Output phase portrait of d. r. c. system #1 without saturation switched for constant 5% overshoot.

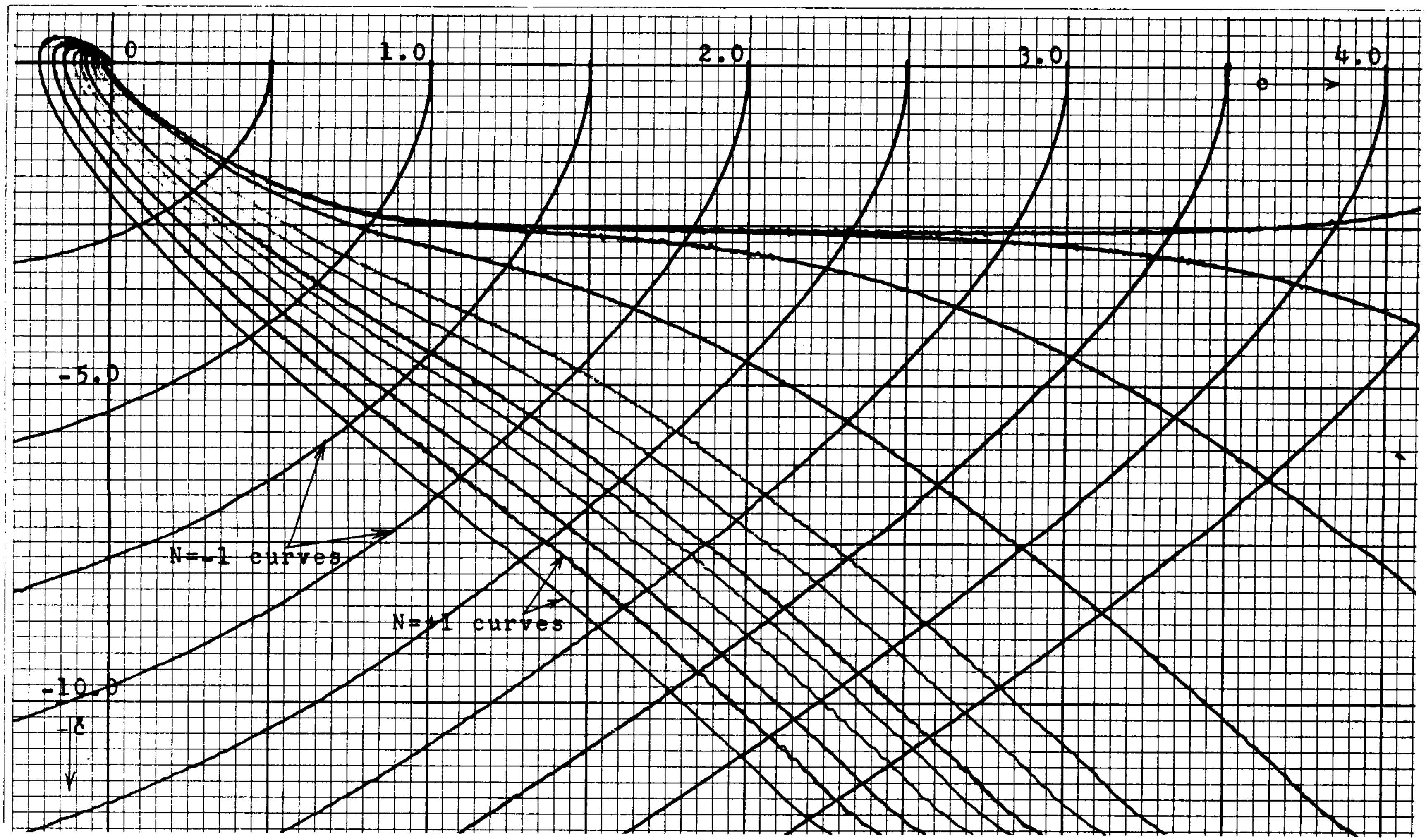


Figure 4.11. Output phase portrait of d. r. c. system #1 with saturation and without switching for N=+1 and N=-1.

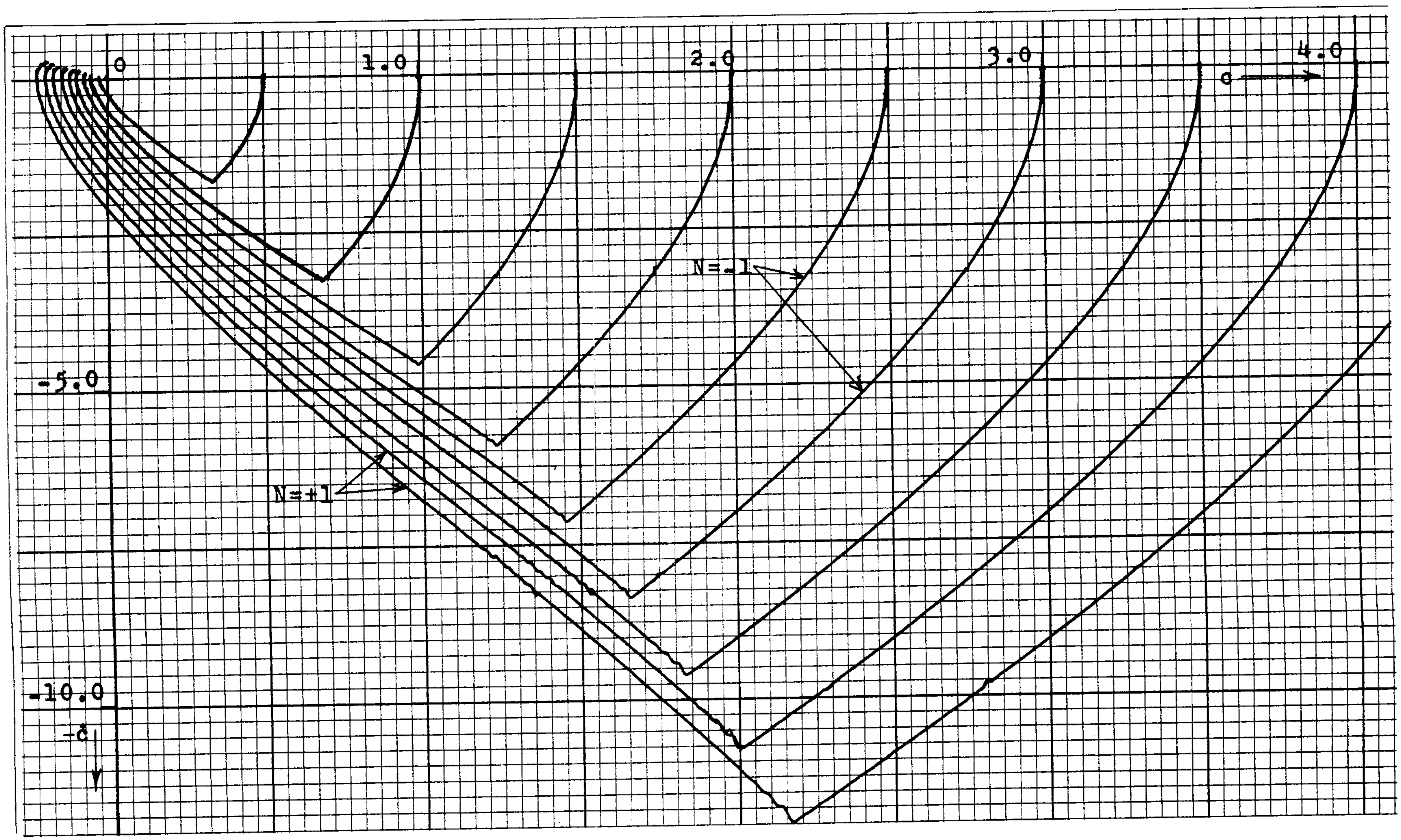


Figure 4.12. Output phase portrait of d. r. c. system #1 with saturation and switched for constant 5% overshoot.

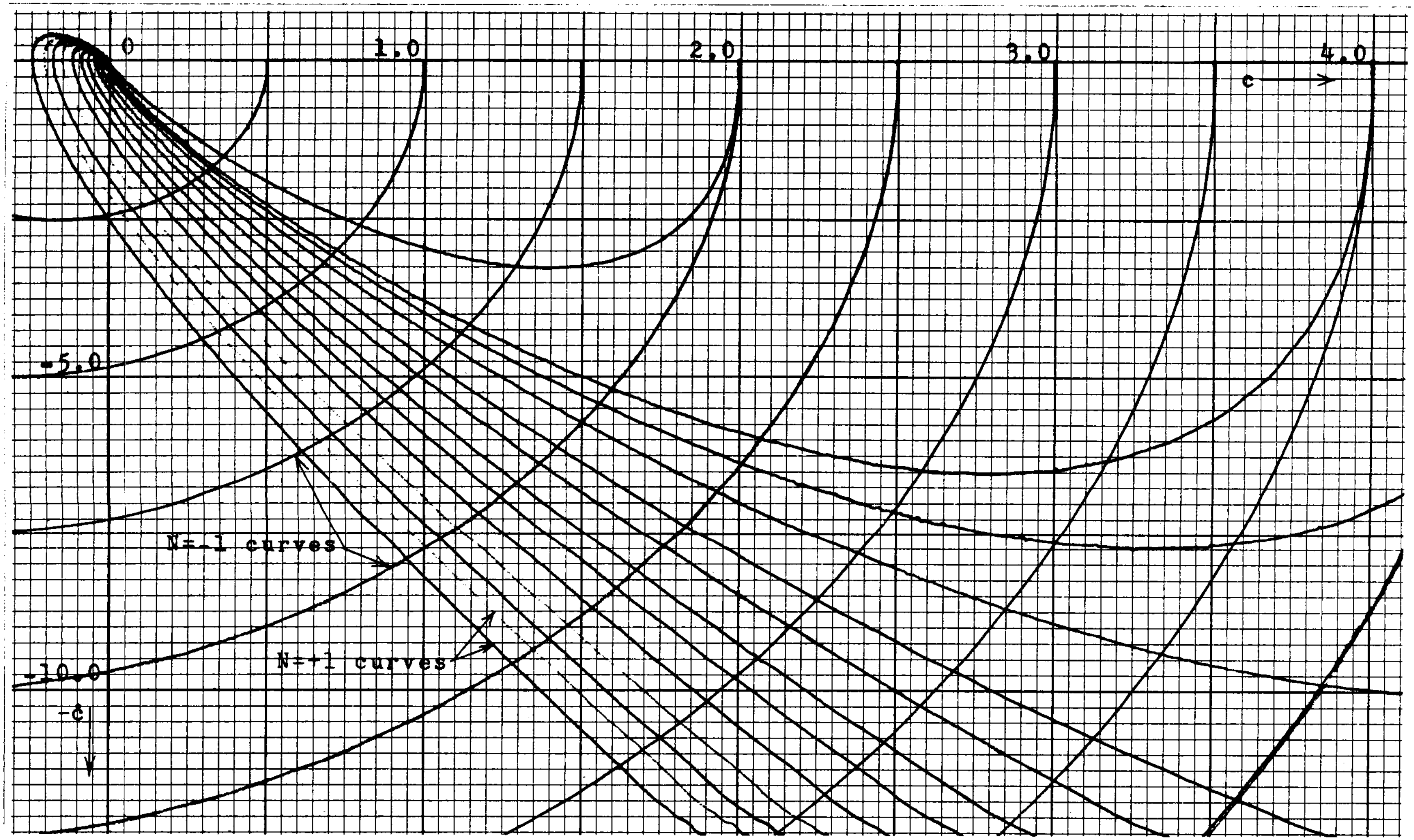


Figure 4.13. Output phase portrait of d. r. c. system #2 without saturation and without switching for N=+1 and N=-1.

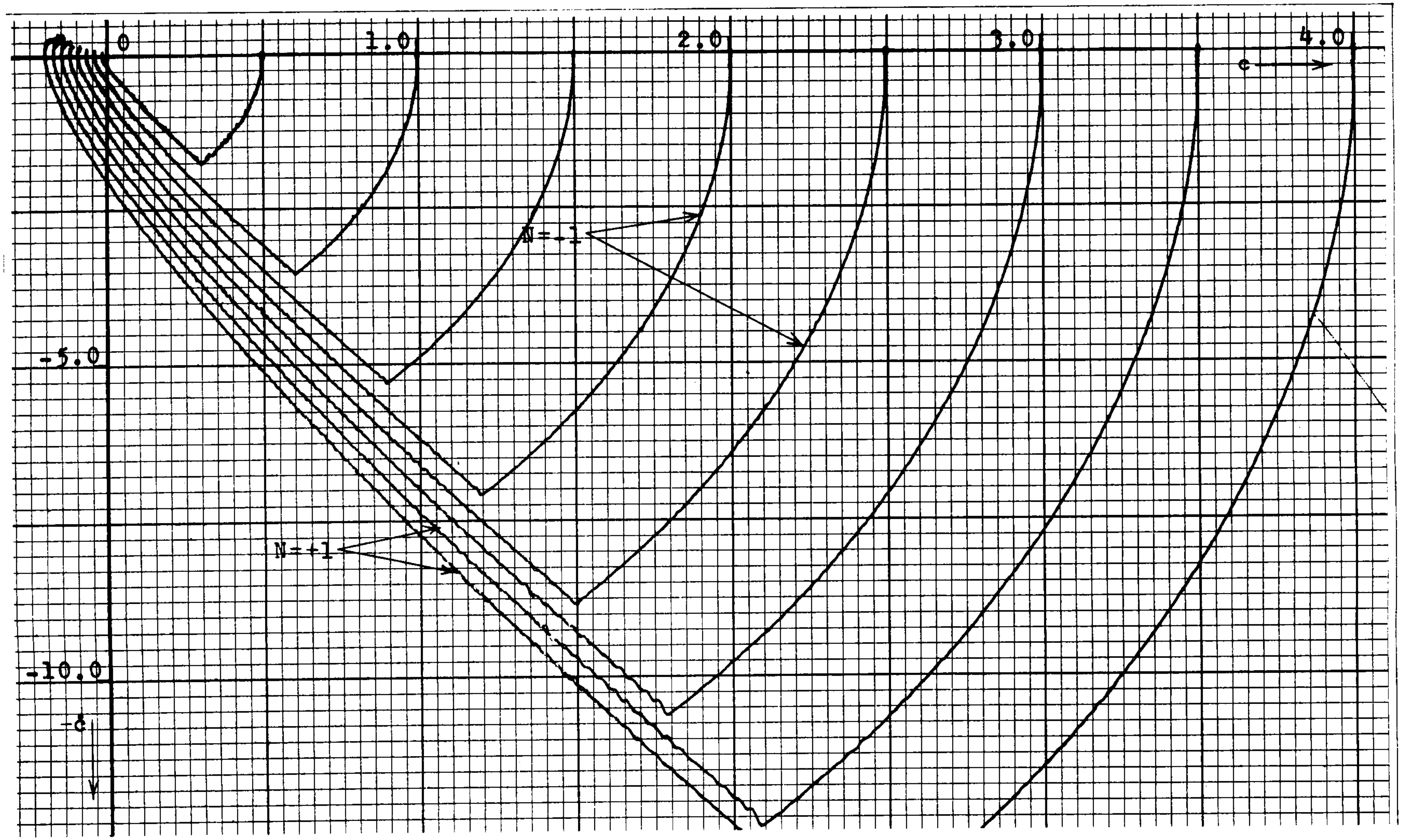


Figure 4.14. Output phase portrait of d. r. c. system #2 without saturation switched for constant 5% overshoot.

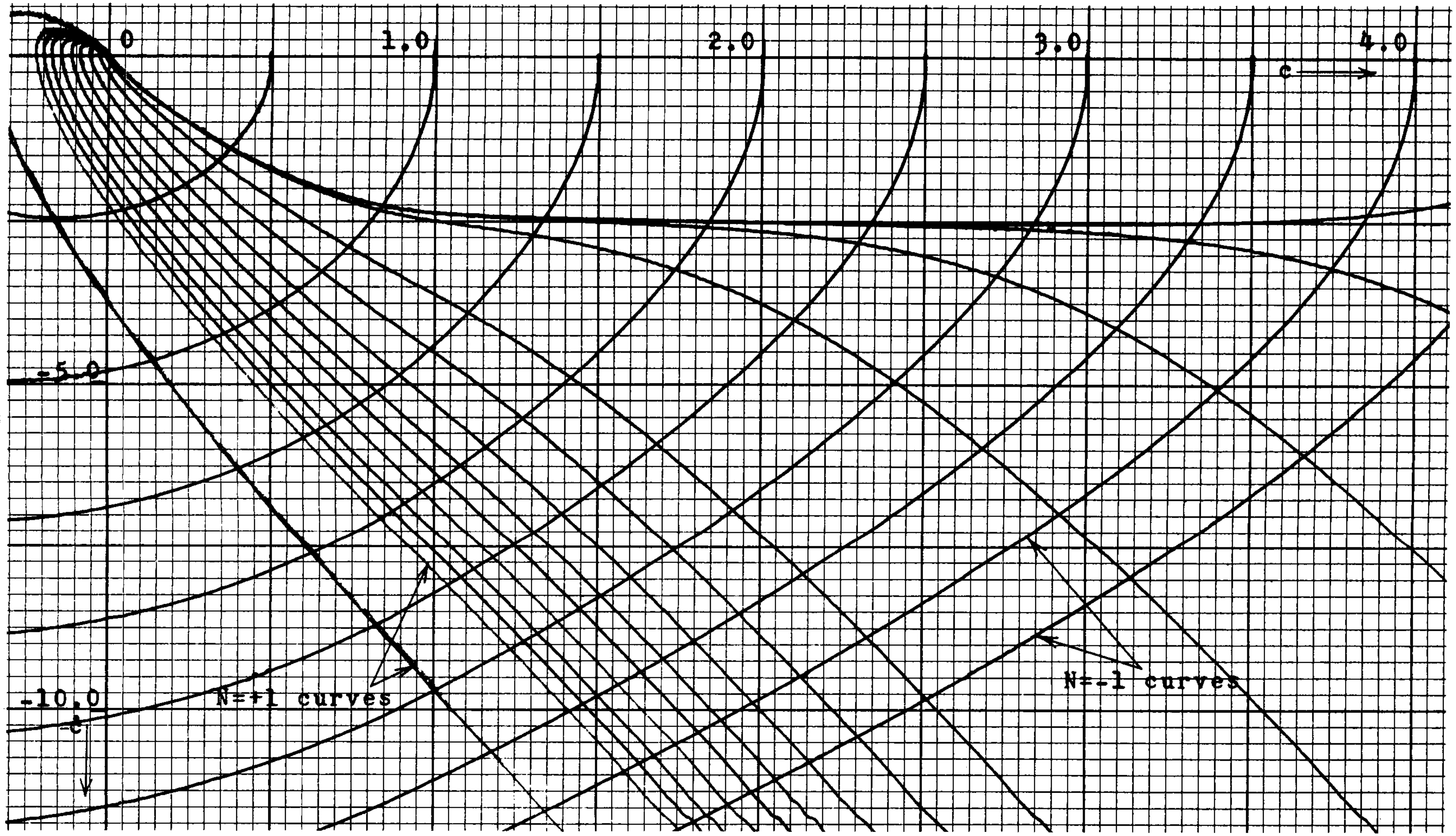


Figure 4.15. Output phase portrait of d. r. c. system #2 with saturation and without switching for N=+1 and N=-1.

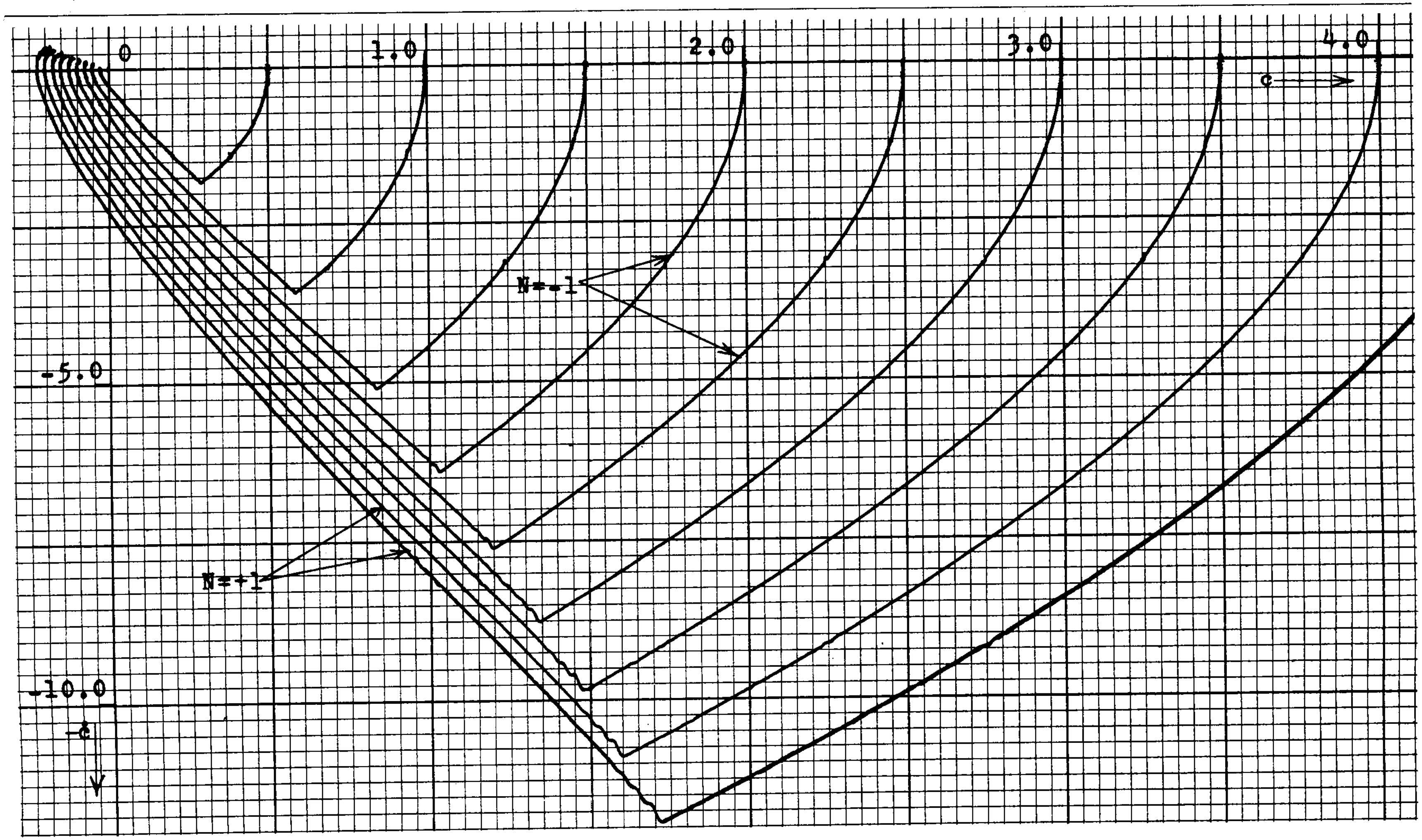


Figure 4.16. Output phase portrait of d. r. c. system #2 with saturation switched for constant 5% overshoot.

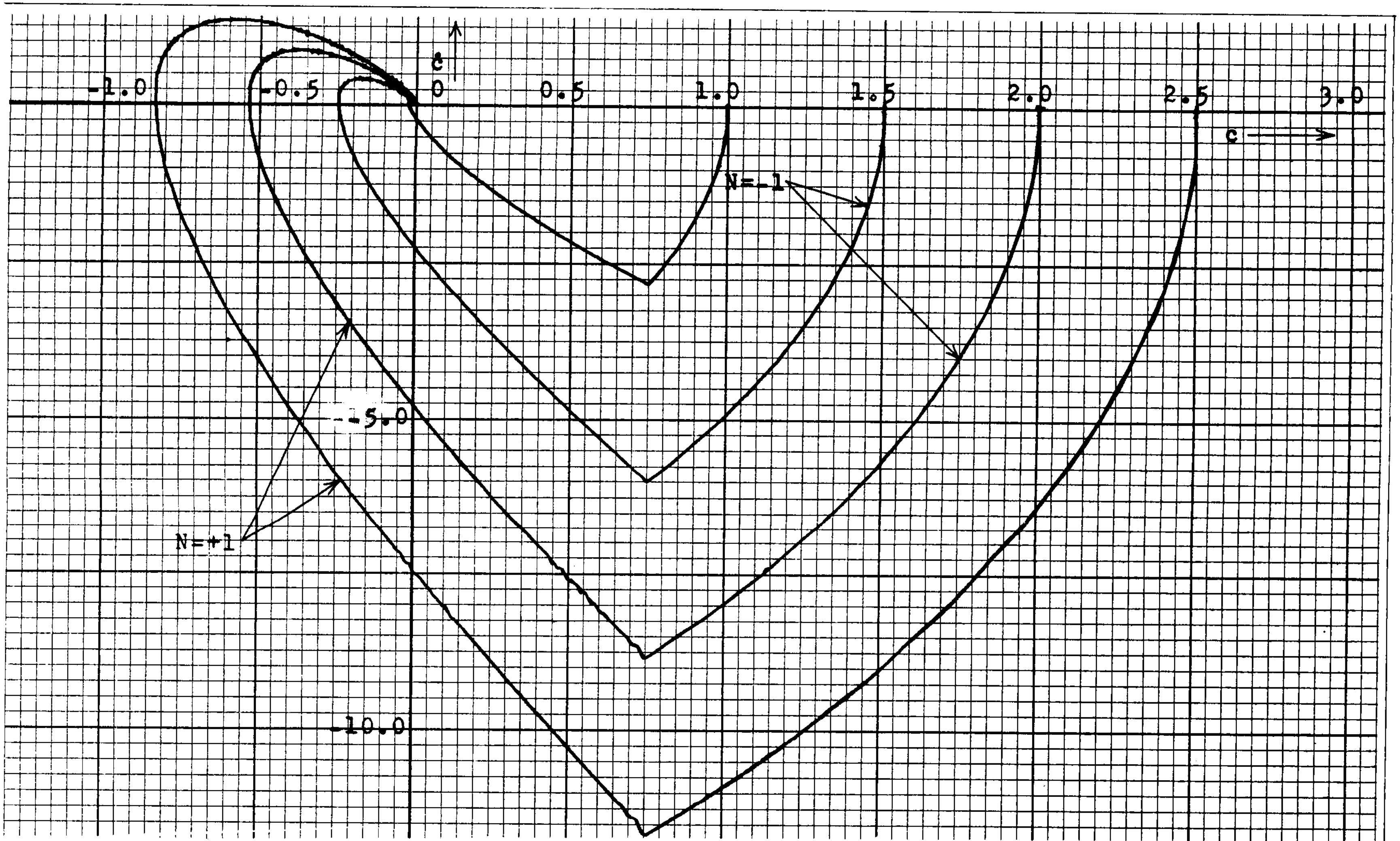


Figure 4.17. Output phase portrait of d. r. c. system #1 without saturation switched at a constant output level of c = 0.75.

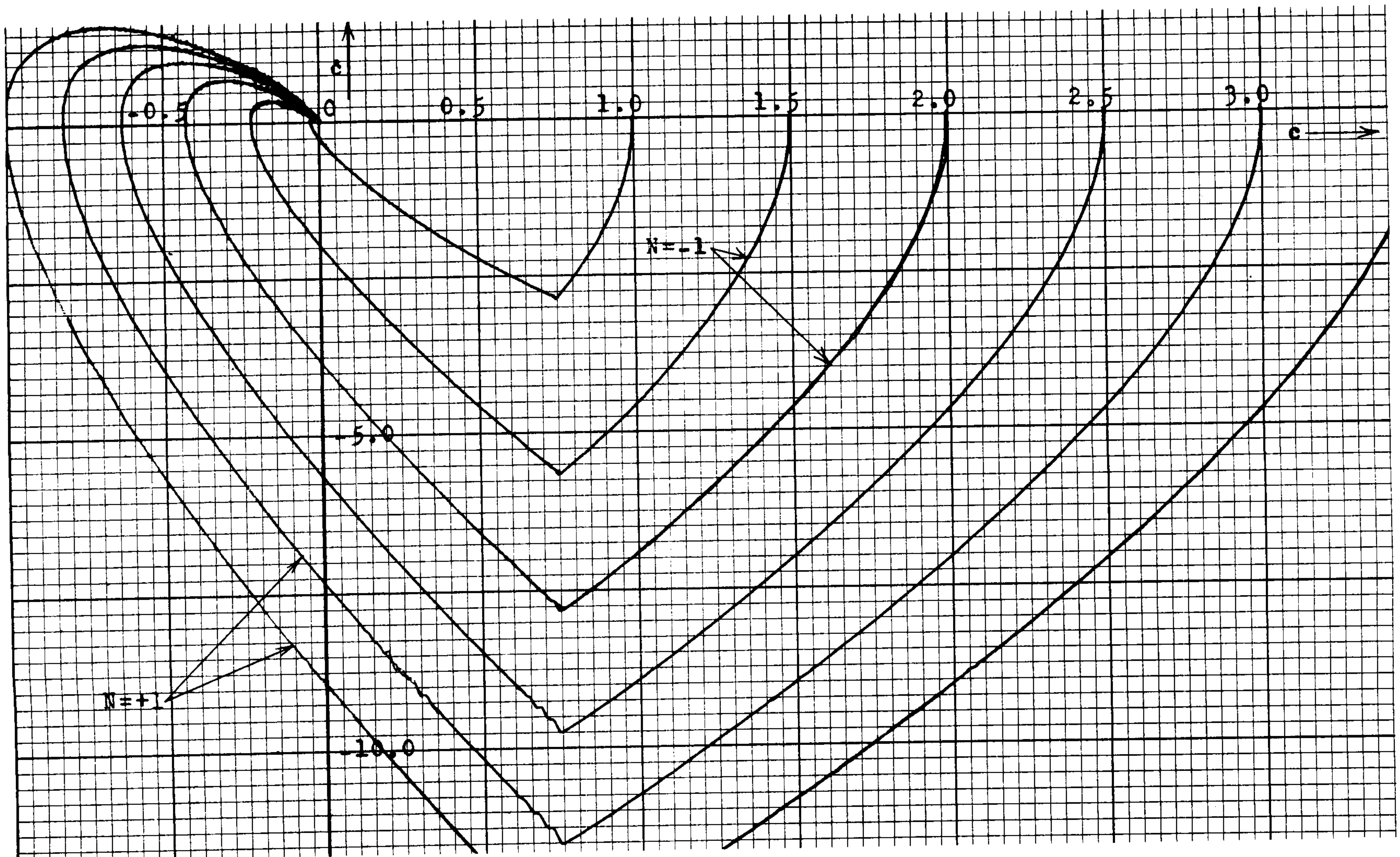


Figure 4.18. Output phase portrait of d. r. c. system #1 with saturation switched at a constant output level of c = 0.75.

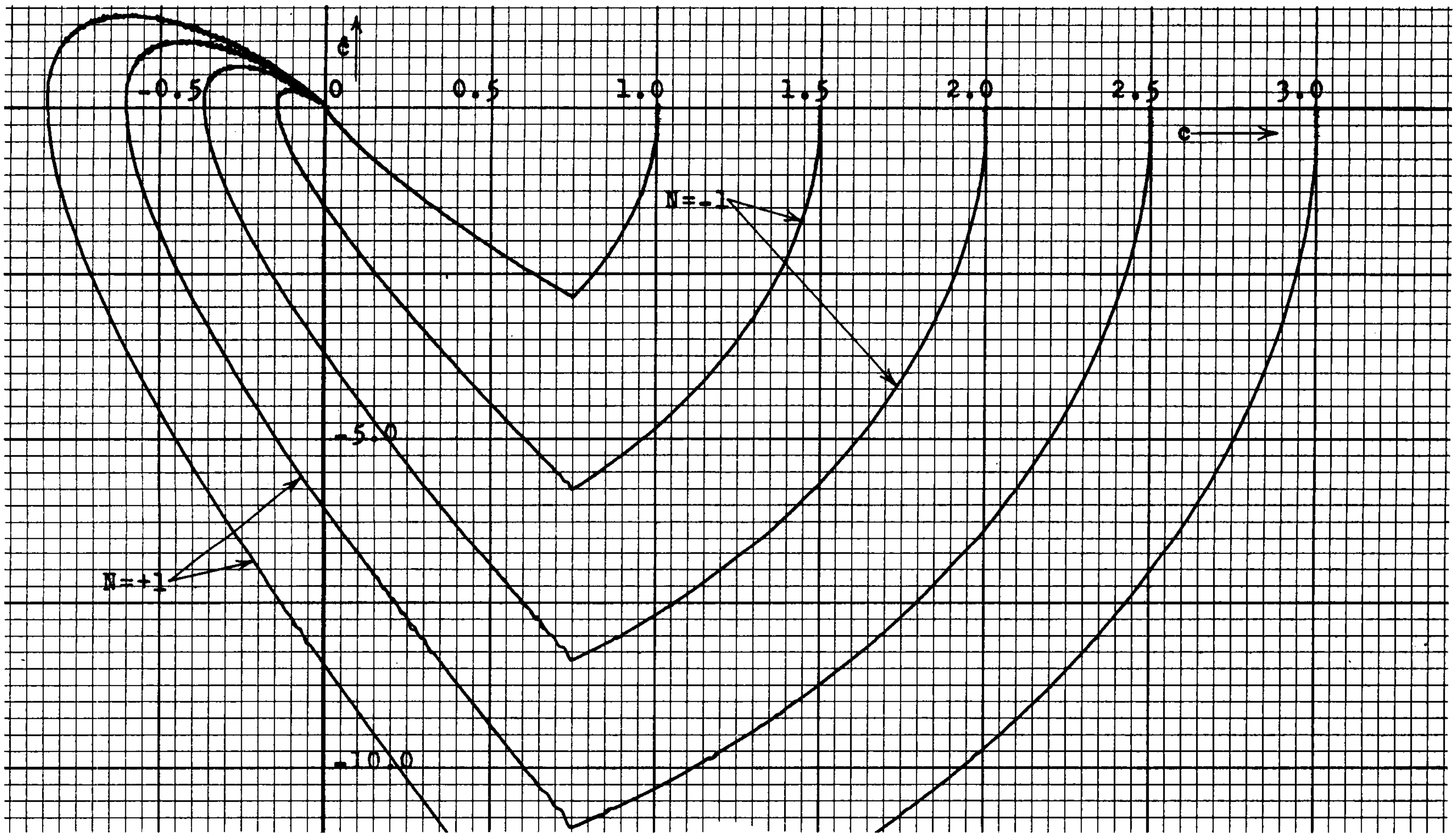


Figure 4.19. Output phase portrait of d. r. c. system #2 without saturation switched at a constant output level of c = 0.75.

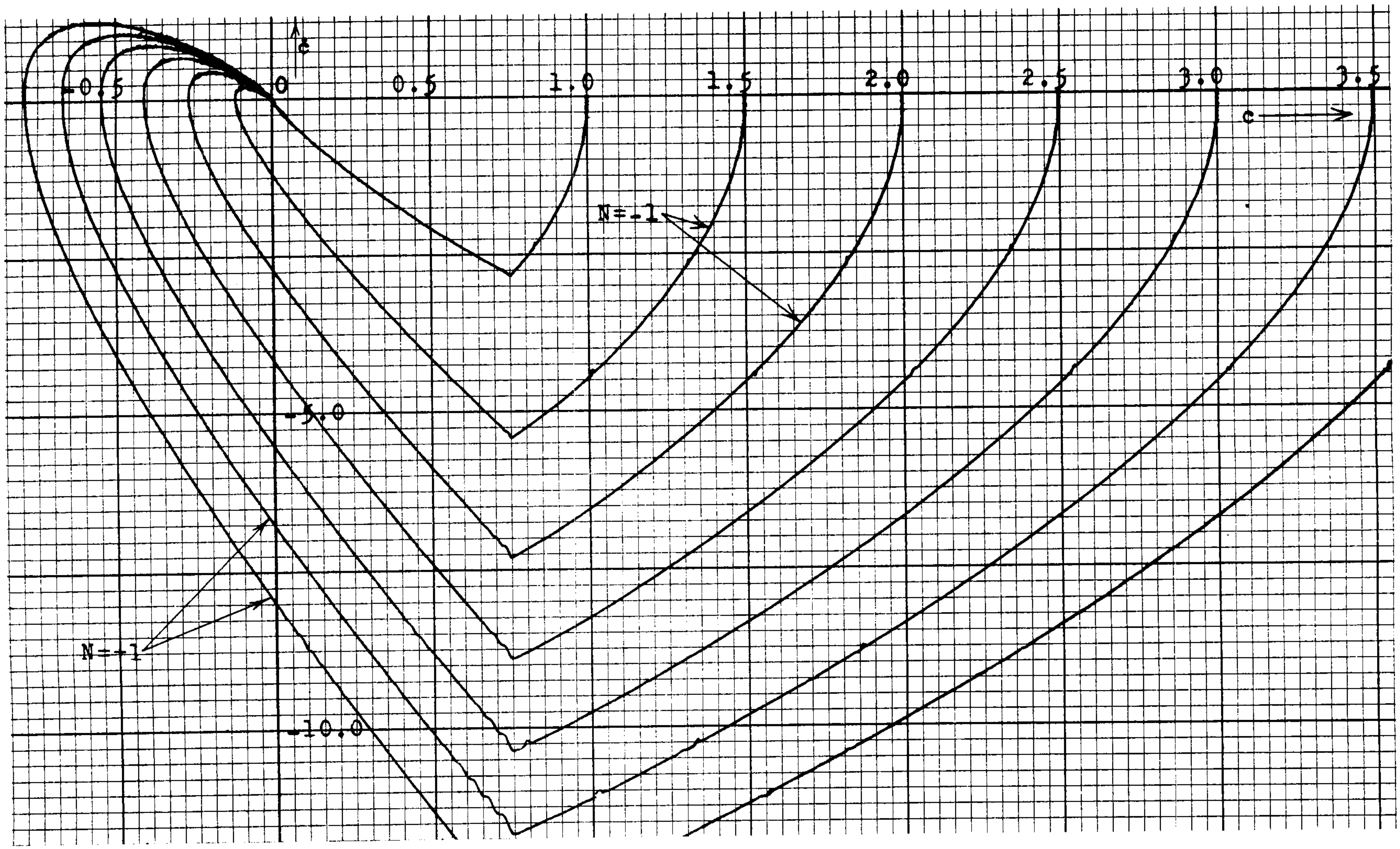


Figure 4.20. Output phase portrait of d. r. c. system #2 with saturation switched at a constant output level of c = 0.75.

II. ANALYSIS OF RESULTS

The results obtained in this study were analyzed to evaluate the relative output performance of each system configuration studied. These results are summarized in Tables 4.1 through 4.8 and in Figures 4.1 through 4.20. Of primary interest was the comparison between the output performance of the two d. r. c. systems and of the bang-bang system. In general, it was found that any comments which applied to system #1 also applied to system #2. The results were analyzed on a system-by-system basis, in the order presented in the data.

The Basic System

The basic system configuration, block-diagrammed in Figure 2.1, was the first studied. This system was chosen because of its relative instability and simplicity. In both its unsaturated and saturated configurations, this system was found to exhibit relatively short rise times and high overshoots, as can be seen in Tables 4.1 through 4.4. The basic system output was found to exhibit rather high output velocities and accelerations, especially in the unsaturated case, because of this relative instability. When saturation was added, rise times increased and overshoots, maximum velocities and maximum accelerations decreased, as would be expected. This effect was most marked at the higher initial output values, as is best illustrated by its effect on the shape of the phase trajectories in Figures 4.1 and 4.2. A need was seen to improve the relative stability of the output of the basic system without adversely affecting the rise time too much.

Basic System with Tachometric Feedback

A commonly-used method of increasing system stability is that of adding a tachometric feedback loop to the driven plant. This method was attempted with the basic system. The basic system with tachometric feedback is block-diagrammed in Figure 2.3. Although the stability of the system was dramatically increased, as evidenced by the near lack of overshoot shown in Tables 4.3 and 4.4, the output rise time was approximately doubled for both system #1 and #2, with and without saturation. This effect is shown in Tables 4.1 and 4.2, but is most effectively illustrated by comparing the trajectories in Figures 4.3 and 4.4 with those in Figures 4.1 and 4.2. As was mentioned in Chapter II, the decrease in area under each trajectory resulting from the addition of tachometric feedback is an indication of increased rise time.

From Tables 4.5 and 4.6 as well as from the phase portraits, it is seen that the maximum velocities were substantially reduced through the addition of tachometric feedback, although the maximum accelerations were unaffected, as shown in Tables 4.7 and 4.8. This lack of effect on maximum accelerations existed because, in each case for each system, the initial acceleration was the maximum value; tachometric feedback has no effect until some finite output rate, or velocity, is built up. Although the addition of tachometric feedback to the basic system had the desirable effect of increasing system stability and reducing oscillation, the price paid in increased rise time was substantial. Neither configuration was deemed to yield satisfactory results.

The Bang-Bang System

The bang-bang system was studied. This configuration could only be logically compared with the saturated cases of the basic configurations since, as pointed out in Chapter II, saturation in the forward-loop amplifier is one of the assumptions made in deriving the bang-bang solution. The bang-bang system configuration is block-diagrammed in Figure 3.2. Output phase portraits of the bang-bang configurations studied are shown in Figures 4.5 through 4.8.

The outputs of the bang-bang systems exhibited the fast rise time of the basic system without any of its resultant overshoot. In fact, the rise times of the bang-bang system were seen, from Tables 4.1 and 4.2 to be even shorter than those of the saturated basic system for low initial output values. This phenomenon occurred because the drive available to the bang-bang system was a constant maximum at all initial output values, and this advantage was most pronounced at the lower initial outputs. As is apparent from the results, tachometric feedback is unnecessary in a bang-bang system except, possibly, to limit maximum output velocities. However, the maximum output accelerations, which are more likely to be a physical consideration than the velocities, were found to be consistently higher in the bang-bang system with tachometric feedback than in the same system without it (see Tables 4.7 and 4.8).

Although the output performance of the bang-bang system was found to be consistently better than that of either the basic system or of the basic system with tachometric feedback, one great drawback to the practical use of this configuration was encountered. At precicely the time the output displacement, velocity and acceleration reached their final state (i.e., zero error for displacement and zero magnitude for both velocity and acceleration), the drive on the system had to be turned off, or limit-cycle operation would result. Moreover, the selection of a switching point for the ideal relay had to be precice, or the final output state could not be reached with one switching. Although the bang-bang system was found to have theoretically superior output responses when compared with the basic configurations, this one physical drawback limits the practicality of this type of control.

Discontinuous Rate Compensated System

Two d. r. c. system schemes were studied. In the first case, the switching point was adjusted for each initial condition such that the overshoot was a constant 5% of the initial value. In the second case, the switching point was maintained at a constant level of output; viz., c=0.75. Because the results obtained in the two cases were substantially different, each case is discussed separately.

Constant 5% Overshoot Case. The output data for the d. r. c. system switched such that the overshoot was always 5% is presented in Tables 4.1 through 4.8 in the column headed "D. R. C. Sys. A". The output phase portraits of this system are shown in Figures 4.10, 4.12, 4.14 and 4.16.

Switching curves were found. The output phase portraits of the system without switching for both positive and negative tachometric feedback are shown in Figures 4.9, 4.11, 4.13 and 4.15. The unswitched curves were used to determine the set of switching points which would result in 5% overshoot regardless of the initial output level. It was found that, for the unsaturated systems, the "switching curve", or locus of switching points, was a straight line from the origin into the fourth quadrant of the phase plane. These lines can be seen by connecting the switching points in Figures 4.10 and 4.14.

The unsaturated system switching curves were found experimentally to be straight lines. This effect could be attributed to the fact that the unsaturated system was piecewise linear. That is, each switched phase trajectory was exactly the same shape as each other trajectory, only magnified or reduced in size according to the initial output level. The switching curves for the saturated systems, seen in Figures 4.12 and 4.16, were curved because the system was no longer piecewise linear, making the shape of each trajectory different from the others.

The time-response of the constant-overshoot system was analyzed. From Tables 4.1 and 4.2, it is seen that the rise times at the higher values of initial output for the constant-overshoot system were less than those of any other system configuration save the basic system without saturation and d. r. c. system B. Moreover, the overshoot of this system never exceeded 5%, whereas in those systems with shorter rise times the overshoots encountered were as high as 40%. The bang-bang system was the only configuration with a combination of shorter rise time and less overshoot than that of the constant-overshoot d. r. c. system, and then only for low values of initial output. Moreover, the switching points for the d. r. c. system were not required to be precicely set, as was the case for the bang-bang system. The rise time advantage of this d. r. c. system over the bang-bang system increased with the magnitude of the input step, or initial output value. The price paid for this performance on the part of the constant-overshoot system is evident from Tables 4.5 through 4.8. The d. r. c. system A exhibited greater output velocity and acceleration magnitudes than any other configuration except d. r. c. system B, with a constant switching level. Except for the velocity and acceleration criteria, which may or may not be a factor in choosing a physical system, the d. r. c. system with constant overshoot probably has the best and most consistent output response of the system configurations studied.

Constant Switching Level Case. The output response data of the d. r. c. system with a constant switching level of $c = 0.75$ is summarized in Tables 4.1 through 4.8 and in Figures 4.17 through 4.20. Although the response of this system to low values of initial output was fairly fast and exhibited reasonable overshoots, this response degenerated rapidly as the system was subjected to steps of higher magnitudes. The response time became very low for higher values of initial output, but the percent overshoot increased very rapidly and the system output was subjected to higher values of maximum velocity and acceleration than any other system configuration studied. Although switching a d. r. c. system at a constant level of output may be physically expedient, it provides poor output response when the magnitude of the input step is much above this switching level, as is evident from the data.

Conclusions

It can be concluded from the analysis made of the data obtained in this study that the overall output time response of the d. r. c. system switched for constant 5% overshoot is superior to that of any of the other system configurations studied. It was found that the basic system, though its response was fast, exhibited a tendency to oscillate and become unstable. Moreover, this system was found to have a very high "settling" time, or time to a final 5% maximum error. When tachometric feedback was added to this system, the problems of potential instability and high overshoot were replaced by the problem of a greatly increased rise time. The bang-bang system, though its time response was deemed satisfactory and a definite improvement over the basic configurations, was found to have a great practical drawback. Namely, the system drive had to be turned off as soon as the output reached its final state. Moreover, it was found that the switching point for the bang-bang system had to be very precisely set, which is physically difficult to realize. The d. r. c. system switched at a constant output level was found to exhibit reasonably good time response at the lower values of input, but this response was found to exhibit high values of overshoot, maximum velocity and maximum acceleration at input values much higher than the switching level.

CHAPTER V

CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

Conclusions are drawn from the results of this investigation and suggestions are made regarding future investigations of time-optimal control of feedback control systems.

I. CONCLUSIONS

It was found early in this investigation that the most efficient as well as most expedient method of studying time-optimization of feedback control systems is through the use of an analog computer and its associated recording equipment. The equipment used in the present investigation, which was found to yield most satisfactory results, consisted of the following: (1) an E. A. I. Model TR-20 analog computer, (2) an E. A. I. Model 1110 x-y plotter, and (3) a Brush Mark 280 strip-chart recorder. This type of investigation requires the use of trial-and-error procedures initially, and an analog computer was found to be very efficient for this type of application. A digital computer was found to have some possibilities, but the additional mathematical manipulations required and the difficulty encountered in generating nonlinearities were among the disadvantages of this method of study.

The mathematical development of a bang-bang time-optimal system was presented in Chapter II, and the performance of this system configuration was compared with that of a discontinuous rate compensated, or d. r. c. system in Chapter IV. A brief exposition of phase plane principles is made in Chapter II. Analog computer programming principles and model-building techniques were presented in Chapter III. The construction of analog computer models of the systems studied here was also presented in Chapter III.

Conclusions are drawn from the data presented in Chapter IV. It was found that, of the system configurations studied, the best overall time response to step inputs within a range of magnitudes was exhibited by a discontinuous rate compensated system switched such that for any magnitude of input the response would overshoot by five percent. The bang-bang configuration of the same system was found to be more severely limited by the effects of saturation on the response time than the five percent overshoot d. r. c. system. A d. r. c. system switched at a constant output level was found to exhibit excessive overshoot, maximum velocity and maximum acceleration when subjected to inputs much above the switching level. Two sets of system parameters were used, making the results of this study applicable to at least two different systems. Since the results obtained from the two parameter sets did not differ substantially, then the conclusions which can be drawn from these results probably apply to a class of systems, and are not limited to one case only.

II. SUGGESTIONS

Suggestions are made for further studies in time-optimal control. Since the results of this study show definite advantages in the use of discontinuous rate compensation in at least a class of second-order systems, it is felt that further study should be made in determining the range of applicability of these results. First, it should be determined for what class or classes of second-order systems with step inputs the results of the present study applies. Next, it should be determined whether these results apply only for step inputs or whether they are valid for ramp, sinusoidal and other inputs. Finally, the study should be extended to third- and higher-order systems. It is also felt that means of generating non-linear switching functions should be investigated, since the use of simple, linear switching curves was found extremely useful in the present study. If high-speed generation of large masses of accurate data is found necessary, the use of sophisticated programming techniques for a digital computer should be investigated. Finally, an attempt should be made to apply formal optimizing techniques to discontinuous rate compensated systems.

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APPENDIX

DIGITAL COMPUTER PROGRAM TO FIND TIME RESPONSE OF BASIC SYSTEM

```
C    BASIC SYSTEM TIME RESPONSE TO UNIT STEP INPUT
        PUNCH 3
        PUNCH 4
    100 READ 1,A,P,B,W,PHI
        DO 200 I=1,61
        FI=I-1
        T=FI*.05
        C=A-(P*EXP(B*T))*(SIN(W*T+PHI))
    200 PUNCH 2,T,C
        PUNCH 5
        PRINT 6
        GO TO 100
      1 FORMAT(5F10.0)
      2 FORMAT(20X,F5.2,15X,F10.6)
      3 FORMAT(22X,1HT,20X,4HC(T))
      4 FORMAT(40H                                        )
       1(40H                                        )
      5 FORMAT(40X35X1H-)
      6 FORMAT(14HCALC. COMPLETE/19HLOAD NEXT DATA SET.)
        END
```